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GROWTH OF COMPOSITE INTEGRAL FUNCTIONS

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In the present paper we study some growth properties of log $T(r, fg)$ relative to $T(r, f)$ and $T(r, g)$ for integral functions $f(z)$ and $g(z)$.

1. INTRODUCTION AND DEFINITIONS

Let $f(z)$ and $g(z)$ be two integral functions. We suppose that $T(r, f)$, $M(r, f)$, $N(r, a, f)$, $\delta(a, f)$, $\delta(a, (z), f)$, $\log^+ x$ etc. bear their usual meanings in the Nevanlinna theory of meromorphic functions (cf. Hayman[2]). Clunie[1] (see also Singh[7]) studied the comparative growths of $T(r, fg)$ with $T(r, f)$ and $T(r, g)$; he showed for transcendental integral functions $f(z)$ and $g(z)$ that $\lim_{r\to\infty} \frac{T(r, fg)}{T(r, f)} = \infty$ and $\lim_{r\to\infty} \frac{T(r, fg)}{T(r, g)} = \infty$. Singh[7] proved some comparative growth properties of log $T(r,fg)$ and $T(r, f)$; also he raised the question of investigating the comparative growth of log $T(r, fg)$ and $T(r, g)$ which he was unable to solve. In the present paper we prove a few theorems on the comparative growths of log $T(r,fg)$ with $T(r, f)$ and, as well as, with $T(r, g)$. Throughout the paper we denote by $f(z)$ and $g(z)$ two integral functions with orders (lower orders) $\rho_f\ (\lambda_f)$ and $\rho_g\ (\lambda_g)$ respectively.

Definition 1—The number $\bar{\lambda}_g$ is said to be the hyper lower order of $g(z)$ if and only if

$$\bar{\lambda}_g = \liminf_{r\to\infty} \frac{\log\log\log M(r, g)}{\log r}.$$

It is clear that $\bar{\lambda}_g \leqslant \lambda_g$.

Definition—2[6]—A function $\rho_g(r)$ is called a proximate order of $g(z)$ relative to $T(r, g)$ if and only if (i) $\rho_g(r)$ is real, continuous and piecewise differentiable for $r > r_0$, (ii) $\lim_{r\to\infty} \rho_g(r) = \rho_g$, (iii) $\lim_{r\to\infty} r \log r\, \rho'_g(r) = 0$, (iv) $\limsup_{r\to\infty} \frac{T(r, g)}{r^{\rho_g(r)}} = 1$.

Proposition 1—For $\delta > 0$ the function $r^{\rho_g + \delta - \rho_g(r)}$ is ultimately an increasing function of r.

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For,

$$\frac{d}{dr}\, r^{\rho_g+\delta-\rho_g(r)} = \{\rho_g + \delta - \rho_g(r) - r\log r\ \rho'_g\ (r)\}$$

$$r^{\rho_g+\delta-1-\rho_g(r)} > 0$$

for all sufficiently large values of r.

2. Theorems and Lemmas

Singh[7] proved a theorem on the estimation of $\limsup\limits_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)}$, which after modification by Zhou[8] takes the following form.

Theorem 1—Let $f(z)$ and $g(z)$ be integral functions of finite orders such that $g(0) = 0$ and $\rho_g < \lambda_f \leqslant \rho_f$. Then $\lim\limits_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)} = 0.$

Here we remark that for the truth of the above theorem the hypothesis $g(0) = 0$ is not essential. In the following we prove a comparative growth property of $\log T(r, fg)$ and $T(r, f)$ under some weaker hypotheses.

Theorem 2—Let $f(z)$ and $g(z)$ be two nonconstant integral functions such that $\lambda_g < \lambda_f \leqslant \rho_f < \infty$. Then $\liminf\limits_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)} = 0.$

Proof: To prove the theorem we need the following lemma.

Lemma 1 (*Theorem* 1, *Niino and Suita*[4])—Let $f(z)$ and $g(z)$ be integral functions. If $M(r, g) > \dfrac{2+\epsilon}{\epsilon} \mid g(0) \mid$ for any $\epsilon > 0$, then we have

$$T(r, fg) < (1 + \epsilon)\, T(M(r, g), f).$$

In particular if $g(0) = 0$, then $T(r, fg) \leqslant T(M(r, g), f)$ for all $r > 0$.

Proof of the Theorem—In the present case for $\epsilon = 1$ and for all large values of r we see that $M(r, g) > \dfrac{2+1}{1} \mid g(0) \mid$. So we obtain from Lemma 1 that for all large values of r

$$T(r, fg) \leqslant 2T(M(r, g), f). \qquad \ldots(1)$$

Since $\lambda_g < \lambda_f$, we can choose $\epsilon\ (> 0)$ such that $\lambda_g + \epsilon < \lambda_f - \epsilon$. Also for all large values of r, $r^{\lambda_f - \epsilon/2} < T(r, f) < r^{\rho_f+\epsilon}$ and for a sequence of values of r tending to infinity $\log M(r, g) < r^{\lambda_g+\epsilon}$.

Now from (1) we get for all large values of r

$$T(r, fg) \leqslant 2T(M(r, g), f) < 2\,\{M(r, g)\}^{\rho_f+\epsilon}$$

and so for all large values of r

$$\log T(r, fg) < \log 2 + (\rho_f + \epsilon) \log M(r, g).$$

Now for a sequence of values of r tending to infinity we get

$$\log T(r, fg) < \log 2 + (\rho_f + \epsilon)\, r^{\lambda_g + \epsilon}$$

$$< \log 2 + (\rho_f + \epsilon)\, r^{\lambda_f - \epsilon}.$$

So for a sequence of values of r tending to infinity we obtain

$$\frac{\log T(r, fg)}{T(r, f)} < \frac{\log 2}{r^{\lambda_f - \epsilon/2}} + \frac{\rho_f + \epsilon}{r^{\epsilon/2}} \text{ and hence } \liminf_{r\to\infty} \frac{\log T(r, fg)}{T(r, f)} = 0.$$

This proves the theorem.

Singh[7] proved the following theorem.

Theorem 3—Let $f(z)$ and $g(z)$ be integral functions of finite orders with $\rho_g > \rho_f$. Then $\limsup_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)} = \infty.$

The analysis of the proof of Theorem 3 shows that the theorem is true, in general, only if $\lambda_f > 0$, which assumption is not explicitly stated in the theorem. The following example also strengthens this comment.

Example 1—Let $f(z) = z$ and $g(z) = e^z$. Then $\rho_f = \lambda_f = 0$ and $\rho_g = 1$, so $\rho_f < \rho_g$. Also $fg(z) = e^z$ and hence $\log T(r, fg) = \log r + O(1)$, $T(r, f) = \log r$, for $r > 1$. Therefore $\limsup_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)} = 1$ which is contrary to Theorem 3.

In the following theorem we see that the conclusion of Theorem 3 can also be drawn even under somewhat relaxed hypotheses.

Theorem 4—Let $f(z)$ and $g(z)$ be two integral functions such that

$0 < \lambda_f < \lambda_g < \infty$. Then $\limsup_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)} = \infty.$

Proof : We know that for $r > 0$ (Niino and Yang[5])

$$T(r, fg) \geqslant \frac{1}{3} \log M\left\{ \frac{1}{8} M\left(\frac{r}{4}, g \right) + o(1), f\right\}. \qquad ...(2)$$

Since λ_f and λ_g are the lower orders of $f(z)$ and $g(z)$ respectively, for given ϵ ($0 < \epsilon < \lambda_f$) and for all large values of r we get $\log M(r, f) > r^{\lambda_f - \epsilon}$ and $\log M(r, g)$

$> r^{\lambda_g - \epsilon}$. So from (2) we get for all large values of r

$$T(r, fg) \geqslant \tfrac{1}{3}\{\tfrac{1}{8} M(r/4, g) + o(1)\}^{\lambda_f - \epsilon}$$

$$\geqslant \tfrac{1}{3}\{\tfrac{1}{9} M(r/4, g)\}^{\lambda_f - \epsilon}$$

which gives for all large values of r

$$\log T(r, fg) \geqslant O(1) + (\lambda_f - \epsilon) \log M(r/4, g)$$

$$\geqslant O(1) + (\lambda_f - \epsilon)(r/4)^{\lambda_g - \epsilon}. \qquad \ldots(3)$$

Also since $\liminf_{r\to\infty} \dfrac{\log T(r, f)}{\log r} = \lambda_f$, it follows that for a sequence of values of r tending to infinity $T(r, f) < r^{\lambda_f + \epsilon}$. Hence for a sequence of values of r tending to infinity we obtain from (3) that

$$\frac{\log T(r, fg)}{T(r, f)} > \frac{O(1)}{r^{\lambda_f + \epsilon}} + (\lambda_f - \epsilon)(r/4)^{\lambda_g - \epsilon} \frac{1}{r^{\lambda_f + \epsilon}}$$

which gives $\limsup_{r\to\infty} \dfrac{\log T(r, fg)}{T(r, f)} = \infty$ because we can choose ϵ $(0 < \epsilon < \lambda_f)$ such that $\lambda_f + \epsilon < \lambda_g - \epsilon$. This proves the theorem.

Now the following three theorems give estimations of the growth of the ratio $\dfrac{\log T(r, fg)}{T(r, g)}$, under different circumstances, as r tends to infinity.

Theorem 5—Let $f(z)$ and $g(z)$ be two nonconstant integral functions such that ρ_f and ρ_g are finite. Then

$$\liminf_{r\to\infty} \frac{\log T(r, fg)}{T(r, g)} \leqslant 3 . \rho_f . 2^{\rho_g}.$$

PROOF : It is well known that Hayman[2], p. 18.

$$T(r, f) \leqslant \log^+ M(r, f) \leqslant 3\, T(2r, f) \qquad \ldots(4)$$

where $r > 0$ and $f(z)$ is an integral function. Also we know for integral functions $f(z)$ and $g(z)$ that for $r > 0$ (cf. Niino and Suita[4])

$$\log M(r, fg) \leqslant \log M(M(r, g), f). \qquad \ldots(5)$$

Since $f(z)$ and $g(z)$ are nonconstant and ρ_f is the order of $f(z)$, we get for all large r and given ϵ (> 0) that

$$T(r, fg) < \log M(M(r, g), f) \leqslant \{M(r, g)\}^{\rho_f + \epsilon}.$$

So for all large r

$$\log T(r, f g) \leqslant (\rho_f + \epsilon) \log M(r, g) \qquad \ldots(6)$$

and hence

$$\liminf_{r\to\infty} \frac{\log T(r, f g)}{T(r, g)} \leqslant (\rho_f + \epsilon) \liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)}.$$

Since ϵ (> 0) is arbitraty, it follows that

$$\liminf_{r\to\infty} \frac{\log T(r, fg)}{T(r. g)} \leqslant \rho_f \liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)}. \qquad \ldots(7)$$

Let $\rho_g(r)$ be a proximate order of $g(z)$ relative to $T(r, g)$. Since $\limsup\limits_{r\to\infty} \dfrac{T(r, g)}{r^{\rho_g(r)}} = 1$, it follows that for all large values of r and for given ϵ $(0 < \epsilon < 1)$ $T(r,g) < (1 + \epsilon)\, r^{\rho_g(r)}$. From (4) we get, on replacement of f by g, for all large values of r, $\log M(r, g) \leqslant 3T(2r, g) < 3(1 + \epsilon)\,(2r)^{\rho_g(2r)}$ and so for all large values of r

$$\log M(r, g) < 3(1 + \epsilon) \frac{(2r)^{\rho_g+\delta}}{(2r)^{\rho_g+\delta-\rho_g(2r)}}, \quad \text{where } \delta\ (> 0) \text{ is arbitrary.}$$

Since $r^{\rho_g+\delta-\rho_g(r)}$ is ultimately an increasing function of r, it follows that for all large r

$$\log M(r, g) < 3(1 + \epsilon) 2^{(\rho_g+\delta)}\, \sigma r^{\rho_g(r)}. \qquad \ldots(8)$$

Again since $\limsup\limits_{r\to\infty} \dfrac{T(r, g)}{r^{\rho_g(r)}} = 1$, for a sequence of values of r tending to infinity we abtain

$$T(r, g) > (1 - \epsilon)\, r^{\rho_g(r)}. \qquad \ldots(9)$$

From (8) and (9) we get for a sequence of values of r tending to infinity

$$\log M(r, g) < 3\, \frac{1 + \epsilon}{1 - \epsilon}\, 2^{\rho_g+\delta}\, T(r, g)$$

which gives $\liminf\limits_{r\to\infty} \dfrac{\log M(r, g)}{T(r, g)} \leqslant 3\, \dfrac{1 + \epsilon}{1 - \epsilon}\, 2^{\rho_g+\delta}$. Since δ (> 0) and ϵ $(0 < \epsilon < 1)$ are arbitary, it follows that

$$\liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)} \leqslant 3.2^{\rho_g}. \qquad \ldots(10)$$

Theorem follows from (7) and (10). This proves the theorem.

Theorem 6—Let $f(z)$ and $g(z)$ be two nonconstant integral functions such that ρ_f and λ_g are finite. Also suppose that there exist integral functions $a_i(z))$ $(i=1,2\ldots, n;\ n \leqslant \infty)$ such that (i) $T(r, a_i(z)) = o\{T(r, g)\}$ as $r \to \infty$ for $i = 1, 2, \ldots, n$ and (ii) $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$. Then

$$\limsup_{r\to\infty} \frac{\log T(r, f\,g)}{T(r, g)} \leqslant \pi.\ \rho_f.$$

PROOF : To prove the theorem we require the following lemma.

Lemma 2[3]—Let $g(z)$ be an integral function with $\lambda_g < \infty$, and assume that $a_i(z)$ $(i = 1, 2, \ldots, n;\ n \leqslant \infty)$ are entire functions satisfying $T(r, a_i(z)) = o\{T(r, g)\}$ then if

$$\sum_{i=1}^{n} \delta(a_i(z), g) = 1 \text{ we have } \lim_{r\to\infty} \frac{T(r, g)}{\log M(r, g)} = \frac{1}{\pi}.$$

Proof of the Theorem—From (6) we obtain for all large values of r and for ϵ (> 0) arbitrary

$$\log T(r, f\,g) \leqslant (\rho_f + \epsilon) \log M(r, g).$$

Hence we get $\limsup_{r\to\infty} \dfrac{\log T(r, f\,g)}{T(r, g)} \leqslant (\rho_f + \epsilon) \limsup_{r\to\infty} \dfrac{\log M(r, g)}{T(r, g)}$

and since ϵ (> 0) is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log T(r, f\,g)}{T(r, g)} \leqslant \rho_f \limsup_{r\to\infty} \frac{\log M(r, g)}{T(r, g)}. \qquad \ldots(11)$$

The theorem follows from (11) and Lemma 2. This proves the theorem.

Note 1 : When, in particular, $a_i(z)$'s are constants the assumption (i) of Theorem 6 is obvious and so it need not be stated explicitly.

Theorem 7—Let $f(z)$ and $g(z)$ be two transcendental integral functions such that

(i) $\rho_g < \infty$ and the hyperlower order of $g(z)$, $\bar{\lambda}_g$ is positive

(ii) $\lambda_f > 0$, and

(iii) $\delta(0, f) < 1$.

Then

$$\limsup_{r\to\infty} \frac{\log T(r, f\,g)}{T(r, g)} = \infty.$$

PROOF : To prove the theorem we require the following lemma.

Lemma 3 (*Theorem* 5, *Niino and Suita*[4])—Let $f(z)$ be a transcendental integral function, $g(z)$ a transcendental integral function of finite order, η a constant satisfying $0 < \eta < 1$, and α a positive number. Then we have

$$T(r, fg) + O(1) \geqslant N(r, 0, fg) > \log \frac{1}{\eta} \left[\frac{N\{M((\eta r)^{1/(1+\alpha)}, g), O, f\}}{\log M((\eta r)^{1/(1+\alpha)}, g) - O(1)} - O(1) \right]$$

as $r \to \infty$ through all values.

Proof of the theorem—Since $\delta(0, f) < 1$, for given $\epsilon > 0$ there exists a sequence of values of r tending to infinity for which $\frac{N(r, 0, f)}{T(r, f)} > 1 - \delta(0, f) - \epsilon > 0$. Hence from Lemma 3 we get for a sequence of values of r tending to infinity

$$T(r, fg) + O(1) \geqslant \log \frac{1}{\eta} \times \frac{(1 - \delta(0, f) - \epsilon) T\{M(\eta r)^{1|(1+\alpha)}, g), f\} - \log M((\eta r)^{1/(1+\alpha)}, g) O(1)}{\log M((\eta r)^{1/(1+\alpha)}, g) - O(1)} \quad \ldots(12)$$

Since $g(z)$ is of finite order ρ_g it follows for given $\epsilon > 0$ and for all large values of r, $\log M(r, g) < r^{\rho_g + \epsilon}$. So from (12) we get for a sequence of values of r tending to infinity

$$T(r, fg) + O(1) \geqslant \log \frac{1}{\eta} \times \frac{(1 - \delta(0, f) - \epsilon) T\{M((\eta r)^{1/(1+\alpha)}, g), f\} - \log M((\eta r)^{1/(1+\alpha)}, g) o(1)}{(\eta r)^{(\rho_g + \epsilon)/(1+\alpha)} \{1 - o(1)\}}.$$

So for a sequence of values of r tending to infinity

$$\log T(r, fg) + O(1) \geqslant O(\log r) + \log T\{M((\eta r)^{1/(1+\alpha)}, g), f\} + \log \left[1 - \frac{\log M((\eta r)^{1/(1+\alpha)}, g) O(1)}{(1 - \delta(0, f) - \epsilon) T\{M((\eta r)^{1/(1+\alpha)}, g), f\}} \right] \ldots(13)$$

Since $f(z)$ is transcendental, $\lim_{r \to \infty} \frac{T(r, f)}{\log r} = \infty$ and so for given positive number N, however large, and for all large values of r $T(r, f) > N \log r$. Therefore, we obtain from (13) for a sequence of values of r tending to infinity

$$\log T(r, fg) \geqslant O(1) - O(\log r) + \log T\{M(\eta r)^{1/(1+\alpha)}, g), f\} + \log \left[1 - \frac{\log M((\eta r)^{1/(1+\sigma)}, g) O(1)}{(1 - \delta(0, f) - \epsilon) N \log M((\eta r)^{1/(1+\alpha)}, g)} \right]$$

(*equation continued on p.* 906)

$$= O(1) - O(\log r) + \log T\{M((\eta r)^{1/(1+\alpha)}, g), f\}$$
$$+ \log\left[1 - \frac{O(1)}{(1 - \delta(0,f) - \epsilon) N}\right]$$

where N is so large that

$$1 - \frac{O(1)}{(1 - \delta(0,f) - \epsilon) N} > 0.$$

Hence, for a sequence of values of r tending to infinity

$$\log T(r, fg) > O(1) - O(\log r) + \log T\{M((\eta r)^{1/(1+\alpha)}, g), f\}. \quad \ldots(14)$$

Since $g(z)$ is of finite positive hyper lower order $\bar{\lambda}_g$, it follows for all large values of r that

$$\frac{\log\log\log M(r, g)}{\log r} \; \tfrac{1}{2} \bar{\lambda}_g.$$

i. e.,

$$\log M(r, g) > \exp(r^{1/2\, \bar{\lambda}_g}). \quad \ldots(15)$$

Again since $f(z)$ is of positive lower order λ_f, we get for all large values of r and for $0 < M < \lambda_f$

$$\log T(r, f) > M \log r. \quad \ldots(16)$$

From (14), (15) and (16) we obtain for a sequence of values of r tending to infinity

$$\log T(r, fg) \geqslant O(1) - O(\log r) + Me\, (\eta r)^{\bar{\lambda}_g/2(1+\alpha)}$$

which gives for a sequence of values of r tending to infinity

$$\frac{\log T(r, g)}{T(r, g)} > O(1) - \frac{O(\log r)}{T(r, g)} + MT(r,g)\; e^{(\eta r)^{\bar{\lambda}_g/2(1+\alpha)}}$$

$$\geqslant O(1) + M \frac{e^{(\eta r)^{\bar{\lambda}_g/2(1+\alpha)}}}{r^{\rho_g + 1}}$$

because $g(z)$ is transcendental and

$$\limsup_{r\to\infty} \frac{\log T(r, g)}{\log r} = \rho_g.$$

This inequality gives

$$\limsup_{r\to\infty} \frac{\log T(r, fg)}{T(r, g)} = \infty.$$

This proves the theorem.

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