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ON L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SUMS

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We introduce here new modified cosine and sine sums

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta(a_j/j)\, k \cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta(a_j/j)\, k \sin kx$$

respectively and study their L^1-convergence. We also deduce results about L^1-convergence of cosine and sine series.

1. INTRODUCTION

Consider cosine and sine series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \qquad \ldots(1.1)$$

$$\sum_{k=1}^{\infty} a_k \sin kx \qquad \ldots(1.2)$$

or together

$$\sum_{k=1}^{\infty} a_k \phi_k(x) \qquad \ldots(1.3)$$

where $\phi_k(x)$ is cos kx or sin kx respectively. Let the partial sum of (1.3) be denoted by $S_n(x)$ and $t(x) = \lim_{n\to\infty} S_n(x)$.

The following results are known :

Theorem A[1] (p. 202)—If $\{a_n\}$ is a quasi-convex null sequence, then (1.1) is a Fourier series of its pointwise limit.

Theorem $B^{2,9}$—If $\{a_n\}$ is a quasi-convex null sequence, then (1.2) is a Fourier series if and only if

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty.$$

If $a_k = o(1)$, $k \to \infty$ and

$$\sum_{k=1}^{\infty} k^2 \left| \Delta^2 \left(\frac{a_k}{k} \right) \right| < \infty \qquad \ldots(1.4)$$

we say that (1.1), (1.2) or (1.3) belongs to the class R.

Kano[4] generalized Theorems A and B by establishing the following results.

Theorem C—If (1.3) belongs to the class R, it is a Fourier series or equivalently, it represents an integrable function.

Theorem D—If $\{a_k\}$ is bounded and quasi-convex, the condition $\sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty$ is equivalent to (1.4).

Concerning the L^1-convergence of (1.2), Kano and Uchiyama[5] proved the following result :

Theorem E—Let $\{a_n\}$ be a bounded, quasi-convex sequence of real numbers. Then (1.2) converges in L if and only if $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$ and $|a_n| \log n \to 0$ as $n \to \infty$.

Rees and Stanojević[7] introduced a cosine sum as $\frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta a_j \cos kx$.

Garrett and Stanojević[3], Ram[6] and Singh and Sharma[8] studied the L^1-convergence of this cosine sum under different sets of conditions on the coefficients a_n.

We introduce here new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{j} \right) k \cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left(\frac{a_j}{j} \right) k \sin kx.$$

The aim of this paper is to study L^1-convergence of $f_n(x)$ and $g_n(x)$ and to obtain, analogous of Theorem E and of the following classical Young-Kolmogorov theorem.

Theorem F^1 (p. 204)—If $\{a_n\}$ is a quasi-convex null sequence, then for the convergence of $\Sigma\, a_n \cos nx$ in the metric space L it is necessary and sufficient that $\lim_{n\to\infty} a_n \log n = 0$.

In what follows, $t_n(x)$ will represent $f_n(x)$ or $g_n(x)$.

2. Results

We prove the following result :

Theorem 1—Let (1.3) be in the class R. Then

$$\lim_{n\to\infty} t_n(x) = t(x) \text{ for } x \in (0, \pi] \text{ and } t \in L(0, \pi] \qquad \ldots(2.1)$$

$$\| t_n - t \| = o(1),\ n \to \infty. \qquad \ldots(2.2)$$

Proof : We will consider only cosine sums as the proof for the sine sums follows the same line. We have

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx$$

$$\times \left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \ldots + \Delta\left(\frac{a_n}{n}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx$$

$$= S_n(x) - \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \qquad \ldots(2.3)$$

where $\tilde{D}_n(x)$ denotes the conjugate Dirichlet kernel. Since $\{a_k\}$ is null, $\lim_{n\to\infty} t_n(x) = t(x)$ for $x \in (0, \pi]$. Theorem C now implies that $t \in L(0, \pi]$.

The relation (2.3) yields

$$t(x) - t_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + \frac{a_{n+1}}{n+1} \tilde{D}'_n(x)$$

$$= \lim_{m\to\infty} \frac{d}{dx} \left(\sum_{k=n+1}^{m} \frac{a_k}{k} \sin kx \right) + \frac{a_{n+1}}{n+1} \tilde{D}'_n(x).$$

Applying Abel's transformation twice, we have

$$t(x) - t_n(x) = \lim_{m\to\infty} \left[\sum_{k=n+1}^{m-1} \Delta\left(\frac{a_k}{k}\right) \tilde{D}'_k(x) + \frac{a_m}{m} \tilde{D}'_m(x) - \frac{a_{n+1}}{n+1} \tilde{D}'_n(x) \right] + \frac{a_{n+1}}{n+1} \tilde{D}'_n(x)$$

$$= \lim_{m\to\infty} \left[\sum_{k=n+1}^{m-2} (k+1)\, \Delta^2\left(\frac{a_k}{k}\right) \tilde{K}'_k(x) + m\,\Delta\left(\frac{a_{m-1}}{m-1}\right) \tilde{K}'_{m-1}(x) - (n+1)\,\Delta\left(\frac{a_{n+1}}{n+1}\right) \tilde{K}'_n(x) + \frac{a_m}{m} \tilde{D}'_m(x) \right]$$

$$= \sum_{k=n+1}^{\infty} (k+1)\, \Delta^2\left(\frac{a_k}{k}\right) \tilde{K}'_k(x) - (n+1)\,\Delta\left(\frac{a_{n+1}}{n+1}\right) \tilde{K}'_n(x)$$

where $\tilde{K}_k(x)$ denotes the conjugate Fejér kernel. Thus

$$\int_{-\pi}^{\pi} |t(x) - t_n(x)|\, dx \leq \sum_{k=n+1}^{\infty} (k+1) \left|\Delta^2\left(\frac{a_k}{k}\right)\right| \int_{-\pi}^{\pi} |\tilde{K}'_k(x)|\, dx + (n+1)\left|\Delta\left(\frac{a_{n+1}}{n+1}\right)\right| \int_{-\pi}^{\pi} |\tilde{K}'_n(x)|\, dx.$$

But, by Zygmund's Theorem[1] (p. 458), we have

$$\int_{-\pi}^{\pi} \left| \widetilde{K}'_k (x) \right| dx = O(k).$$

Moreover,

$$\left| \Delta \left(\frac{a_{n+1}}{n+1} \right) \right| = \left| \sum_{k=n+1}^{\infty} \Delta^2 \left(\frac{a_k}{k} \right) \right|$$

$$\leqslant \sum_{k=n+1}^{\infty} \frac{k^2}{k^2} \left| \Delta^2 \left(\frac{a_k}{k} \right) \right|$$

$$\leqslant \frac{1}{(n+1)^2} \sum_{k=n+1}^{\infty} k^2 \left| \Delta^2 \left(\frac{a_k}{k} \right) \right|$$

$$= o \left(\frac{1}{(n+1)^2} \right)$$

by the given hypothesis. Thus, it follows that

$$\int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx = O \left(\sum_{k=n+1}^{\infty} (k+1)^2 \left| \Delta^2 \left(\frac{a_k}{k} \right) \right| \right) + o(1)$$

$$= o(1) \qquad \text{..(2.4)}$$

by (1.4). This proves (2.2) and the Theorem 1 is proved.

3. Deductions

(i) If (1.3) belongs to the class R, then $\| t - S_n \| = o(1) \ (n \to \infty)$ if and only if $| a_{n+1} | \log n = o(1), \ n \to \infty$.

Proof : We prove this result for cosine series only, the proof for sine series being similar. Using (2.5), we get

$$\int_{-\pi}^{\pi} | t(x) - S_n(x) | \, dx = \int_{-\pi}^{\pi} | t(x) - t_n(x) + t_n(x) - S_n(x) | \, dx$$

$$\leqslant \int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx$$

$$+ \int_{-\pi}^{\pi} | t_n(x) - S_n(x) | \, dx$$

(*equation continued on p.* 913)

$$= \int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx$$

$$+ \int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

and

$$\int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx = \int_{-\pi}^{\pi} | t_n(x) - S_n(x) | \, dx$$

$$\leqslant \int_{-\pi}^{\pi} | t(x) - S_n(x) | \, dx$$

$$+ \int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx.$$

Since

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} | t(x) - t_n(x) | \, dx = 0$$

by our Theorem 1 and

$$\int_{-\pi}^{\pi} \left| \frac{a_{n+1}}{n+1} \widetilde{D}'_n(x) \right| dx$$

behaves like $| a_{n+1} | \log n$ by Zygmund's Theorem, cited above, for large n, the conclusion of the corollary (i) follows.

(ii) We deduce now Theorem E of Kano and Uchiyama as follows :

The condition $| a_{n+1} | \log n = o(1)$, $n \to \infty$ and Theorem D imply that the sine series (1.2) belongs to the class R. The 'if part' now follows from deduction (i). The proof of the 'only if part' is the same as given by Kano and Uchiyama[5].

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