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A NOTE ON THE RADICALS IN A NORMED NEAR-ALGEBRA

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In an earlier paper[4] normed near-algebras were defined and it was shown that the set of all regular elements form an open set. In this paper, radical like objects $J(B)$ and $D(B)$ are introduced and their properties are examined.

1. Through out this paper we follow the same definitions and notations as in a earlier paper[4]. However for convenience sake we briefly recall, the following definitions.

Definition 1 — A (right) near-algebra B over a field F is a linear space over F on which a multiplication is defined such that (i) B forms a semigroup under multiplication, (ii) multiplication is right distributive with respect to addition, (iii) $\alpha\,(xy) = (\alpha x)\,y$ for all $x, y \in B$ and $\alpha \in F$.

Definition 2 — A near-algebra B over the real or complex numbers is called a normed near-algebra provided there is associated with each $x \in B$ a real number $\|x\|$, called the norm of x, with the following properties:

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$ (the additive identity of B),

(ii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in B$,

(iii) $\|\alpha x\| = |\alpha|\,\|x\|$, α a real or complex number,

(iv) $\|xy\| \leq \|x\|\,\|y\|$, for all $x, y \in B$,

(v) If B has an identity e, then $\|e\| = 1$.

Through out this paper we assume that B is a normed near-algebra such that the property:

(vi) $\|xy - xz\| \leq \|x\|\,\|y - z\|$, for all $x, y \in B$ holds in B. And further we assume that

(i) B is complete with respect to the norm defined as above.

(ii) B has an identity e,

(iii) For every $x \in B$, $0.\,x = x.\,0 = 0$

As in the case of near-rings (cf. Pilz[3]) we define a B-space, B-subspace and a near-algebra ideal. The definition of a B-space follows from Brown[2].

Definition 3 — A linear space V over a field F is called a B-space if there exists a mapping $(x, v) \to x.\,v$ from $B \times V$ into V such that for all $x, y \in B$, $v \in V$ and $\delta \in F$,

(i) $(x + y)\,v = xv + yv$,

(ii) $(xy)v = x(yv)$,

(iii) $(\delta x)\, v = \delta\, (xv)$,

(iv) $ev = v$, if B has an identity e, and $x \in B$, $\bar{0} \in V : x\bar{0} = \bar{0}$.

Definition 4 – A subspace R of B-space V with $BR \subseteq R$ is called a B-subspace of V.

As for near-rings we have near-algebra ideals which were introduced by Brown[2]. We denote the near-algebra ideal by NA-ideal.

Definition 5 – A subset I of B is called an NA-ideal if it satisfies the following conditions:

(i) $(I, +)$ is a linear subset of B,

(ii) $ix \in I$ for all $x \in B$, $i \in I$,

(iii) $x\,(i + y) - xy \in I$, for every $x, y \in B$, $i \in I$.

The subset I with the conditions (i) and (ii) is called a right NA-ideal of B, and with (i) and (iii) is called a left NA-ideal of B.

Due to the absence of one of the distributive laws in a near-algebra, one has to distinguish between left NA-ideals and B-subspaces. As in the case of near-rings we have two types of modular left ideals. Since we are assuming that the normed near-algebra B has an identity e, every left NA-ideal is modular.

Definition 6 – A left NA-ideal of B is said to be (i) of type (0), if it is a left NA-ideal which is maximal as a left NA-ideal and (ii) of type (2), if it is left NA-ideal which is maximal as a B-subspace.

With these definitions we have two types of radical like objects, namely, $J(B)$ = Intersection of all maximal left NA-ideals of type (2) and $D(B)$ = Intersection of all maximal left NA-ideals of type (0).

It is easy to see that $D(B) \subset J(B)$.

The following Lemma (1) need not be true in a general normed near-algebra but we prove this using the property:

$$\|xy - xz\| \leq \|x\|\, \|y - z\|$$

of the normed near-algebra B.

Lemma 1 – Let B be a normed near-algebra. If L is a B-subspace (left NA-ideal) of B, then $\bar{L}$ (the closure of L) is also a B-subspace (left NA-ideal) of B.

Proof : Let L be a B-subspace of B. Then $\bar{L} = \{x \in B/$there exists a sequence $\{x_k\}$ in L such that $x_k \to x$ in $B\}$. For every $x, y \in B$ and $x \in \bar{L}$, $\|yx_k - yx\| \leq \|y\|\, \|x_k - x\|$ (by property vi). But as $k \to \infty$, $\|x_k - x\| \to 0$, which implies that $yx_k \to yx$. This shows that $yx \in \bar{L}$. Hence $B\bar{L} \subseteq \bar{L}$. Similarly it can be shown that $\bar{L}$ satisfies all the conditions of left NA-ideal.

Definition 7 – An element $r \in B$ is said to be left (right) regular if there exists an element $s \in B$ such that $sr = e$ $(rs = e)$..

In an earlier paper[4] it has been shown that G, the set of all regular elements is open in B. Further all the operations are continuous in B. This forces us to state the following theorem.

Theorem 1 — Let B be a normed near-algebra. If L is either a maximal B-subspace or a maximal left NA-ideal of B, then L is a proper closed set.

Proof : Since G is open in B and $L \cap G = \phi$, then we have $L \neq B$. By Lemma 1, we have $L = \bar{L}$. This implies that L is closed.

The concepts of quasi-regular elements and quasi-regular B-subgroups were introduced for topological near-rings by Beidlemen and Cox[1]. We define the same for normed near-algebras.

Definition 8 — Let B be a normed near-algebra. An element $r \in B$ is said to be quasi-regular if $(e - r)$ is left invertible.

Definition 9 — Let R be a B-subspace of B. Then R is said to be a Quasi-regular B-subspace if each element of R is quasi-regular.

As a normed linear space is connected and so is a normed near-algebra, we can state the following.

Theorem 2 — In a normed near-algebra B, $J(B)$ and $D(B)$ are nowhere dense.

Proof : By the definitions of $J(B)$ and $D(B)$, it is clear that they are proper closed sets of B. Suppose that interior of $J(B)$ is nonempty. Then there exists a nonempty open set $V \subset J(B)$. Let $g \in V$ then for every $f \in J(B)$, $f - g + V \subset J(B)$, thus $J(B)$ contains a neighbourhood of each of its elements. Hence $J(B)$ is open and closed, contradicting that B is connected. Therefore $J(B)$ is nowhere dense.

Similarly we can prove that $D(B)$ is nowhere dense.

In fact it can be shown that any proper B-subspace is nowhere dense. From this it is clear that $J(B)$ and $D(B)$ cannot contain all quasi-regular elements, as the set of all quasi-regular elements form an open set. However the following results can be stated. The proofs are omitted, as they can be proved as in the case of topological near-rings[1].

Theorem 3 — In a normed near-algebra B, $J(B)$ contains all quasi-regular B-subspaces.

Theorem 4 - The intersection of all maximal B-subspaces is quasi-regular and is contained in $J(B)$.

Corollary 1 — Let R be the intersection of all maximal B-subspaces of B. Then R contains all quasi-regular left NA-ideals of B.

Corollary 2 — $J(B)$ contains all quasi-regular left NA-ideals of B.

2. *Definition* 10 — An element x of B is said to be topologically nilpotent if the sequence $\{x^n\}$ has limit zero.

It can be seen that, every element with $\|x\| < 1$ is topologically nilpotent, hence neither $J(B)$ nor $D(B)$ contain all topologically nilpotent elements, for $0 < |\alpha| < 1$, αe is topologically nilpotent but does not belong to either $J(B)$ or $D(B)$. However $J(B)$ contains all topologically nilpotent B-subspaces.

Definition 11 — A B-subspace Δ is said to be (i) topologically nilpotent if $\{\Delta^n\}$ tends to zero (ii) nilpotent if there is a positive integer n such that $\Delta^n = 0$.

Theorem 5 — A B-subspace Δ is topologically nilpotent implies that Δ is nilpotent.

PROOF: Suppose Δ is topologically nilpotent. Consider the open set $U = \{x/\|x\| < 1\}$ of B. Then there exists a positive integer n such that $\Delta^n \subset U$. For every $c_1, c_2, \ldots, c_n$ in Δ, $c_1 c_2 \ldots c_n \in U$. For any scalar $\alpha \neq 0$, $(\alpha c_1)\, c_2 \ldots c_n \in U$, since Δ is a subspace of B. Therefore $\|\alpha c_1 c_2 \ldots c_n\| < 1$ which implies that $\|c_1 c_2 \ldots c_n\| < 1/|\alpha|$. Taking limits as $|\alpha|$ tends to infinity we have that $c_1 c_2 \ldots c_n = 0$, that is $\Delta^n = 0$.

Theorem 6 — $J(B)$ contains all nilpotent B-subspaces.

PROOF: Suppose K is a nilpotent B-subspace of B, that is $K^n = 0$. Assume that $K^{n-1} \neq 0$. Suppose $K \not\subset J\,(B)$. Then there exists a maximal left NA–ideal L such that $K \not\subset L$. Therefore $B = L + K$. Then for $\lambda \in L$, $x \in K$, $1 = \lambda + x$. We have $K^{n-1} = K^{n-1}\,(\lambda + x) - K^{n-1}\,x \in L$, since $K^{n-1}\,x = 0$. Therefore $K^{n-1} \subset L$. Now $K^{n-2} = K^{n-2}\,(l + x) - K^{n-2}\,x$, since $K^{n-1} \subset L$, we have $K^{n-2}\,x \subset L$. That is $K^{n-2} \subset L$. Proceeding like this we have $K \subset L$ which is a contradiction. Hence $K \subset J\,(B)$.

Corollary 3 — $J(B)$ contains all topologically nilpotent B-subspaces.

PROOF: Since any topologically nilpotent B-subspace is nilpotent, the result follows.

Theorem 7 — If B is locally compact, then every quasi-regular B-subspace is nil.

PROOF: Suppose M is a quasi-regular B-subspace of B. Let $x \in M$. Since B is locally compact, B is finite dimensional. Hence the sequence $Bx \supset Bx^2 \supset \ldots$. must terminate. So there exists a positive integer n such that $Bx^n = Bx^{n+1}$ that is $x^n \in Bx^{n+1}$. This implies $x^n = bx^{n+1}$, for some $b \in B$. Then $(e - bx)x^n = 0$. Since $e - bx$ is invertible, $x^n = 0$. Therefore x is nilpotent. Hence M is nil.

Theorem 8 — The distance between the identity e and $J(B)$ is one.

PROOF: The Proof is is clear, since for any $x \in B$, with $\|e - x\| < 1$ is regular, that is $d(e, J(B)) \geq 1$, but $0 \in J(B)$ and $d(e, 0) = \|e\| = 1$.

Theorem 9 — If $0 \neq x \in B$ is nilpotent then $\|e - x\| \geq 1$.

PROOF : Suppose x is nilpotent, this implies that x is not regular. Then $\|e - x\| \geq 1$, since $\|e - x\| < 1$ implies that x is regular.

But without using this property which is based on the inequality $\|xy - xz\| \leq \|x\|\,\|y - z\|$, we can directly observe that if x is nilpotent, then $\|e - x\| \geq 1$. Since $0 \neq x$ is nilpotent there exists a positive integer n such that $x^n = 0$, $x^{n-1} \neq 0$. We have

$$\begin{aligned} \|x^{n-1}\| &= \|x^{n-1} - x^n\| \\ &= \|(e - x)\, x^{n-1}\| \\ &\leq \|e - x\|\,\|x^{n-1}\| \end{aligned}$$

That is $(\|e - x\| - 1)\|x^{n-1}\| \geq 0$.

Therefore $\|e - x\| \geq 1$.

Corollary 4 — If Δ is nilpotent B-subspace of B then $d\ (e, \Delta) \geq 1$.

Beidleman and Cox[1] have introduced the right boundedness in topological near-rings. In the similar way right boundedness and left boundedness in the normed near-algebras are defined as follows:

Definition 12 — A subset Δ of B is said to be right bounded (left bounded) provided that for any neighbourhood U of 0, there is an open set V such that $V\Delta \subseteq U\ (\Delta V \subseteq U)$.

It can be seen that right boundedness coincides with boundedness in a normed near-algebra with identity. However with regard to left boundedness we can state the following result.

Theorem 10 — If Δ is a B-subspace which is left bounded then Δ is a left zero divisor. In particular there exists an open set V such that $\Delta V = 0$.

Proof : Let Δ be a left bounded B-subspace of B. That is there exists an open set V such that $\Delta V \subset S_\epsilon\ (0)$, where $S_\epsilon\ (0)$ is an open sphere with centre at 0, and radius ϵ. $F\Delta V \subset S_\epsilon\ (0)$ is clear. Therefore $\alpha bx \in S_\epsilon\ (0)$ for any scalar α, for any $b \in \Delta$, $x \in V$. So we have $\|bx\| < \epsilon/_{|\alpha|}$. This tends to zero as $|\alpha| \rightarrow \infty$ which implies that $bx = 0$. Therefore $\Delta V = 0$.

Corollary 5 — Any normed near-algebra b is not left bounded.

Proof : If B is left bounded, then there exists an open set U such that $BU = 0$. This implies that $U = \{0\}$. But the single point set $\{0\}$ is closed but not open. Hence B is not left bounded.

S.3 An example of an infinite dimensional normed near-algebra was given in a earlier paper[4]. Here the construction of 2-dimensional normed near-algebra is given.

Example — Let B be a two dimensional linear space with basis $\{e, w\}$. As any two norms will be equivalent, the norm can be taken as $\|\alpha e + \beta w\| = |\alpha| + |\beta|$. If B were to be a normed near-algebra (right) with identity e, the multiplication will will be determined if $w.x$ is specified. Let $w.y = f(y)e + g(y)\ w$.

$$\left.\begin{aligned} \text{Then } x.y &= [\alpha a + \beta f\ (y)]\ e + [\alpha b + \beta g\ (y)]\ w, \\ \text{where } x &= \alpha e + \beta w \text{ and } y = ae + bw \end{aligned}\right\} \quad \ldots(1)$$

The associativity of multiplication leads to the following equations:

$$\left.\begin{aligned} f\ (yz) &= f\ (y)\ p_1\ (z) + g\ (y)\ g\ (z) \\ f\ (yz) &= f\ (y)\ p_2\ (z) + g\ (y)\ g\ (z) \end{aligned}\right\} \quad \ldots(2)$$

where p_1 and p_2 are projections on the first and second coordinates.

Further the condition that e acts as identity gives the equations

$$g\ (e) = 1,\ f\ (0) = g\ (0) = f\ (e) = 0 \quad \ldots(3)$$

As we are assuming that

$$\|xy - xz\| \leq \|x\|\ \|y - z\|,$$

we have

$$|f\ (y) - f\ (z)| + |g\ (y) - g\ (z)| \leq |p_1\ (y) - p_1\ (z)| + |p_2\ (y) - p_2\ (z)| \quad \ldots(4)$$

Equation (4) shows that the functions f, g are continuous with respect to the norm. So for every pair of functions $f, g : B \to F$ satisfying the above conditions, we have a two dimensional normed near-algebra over the field F. Thus this characterizes all 2-dimensional normed near-algebras.

If $f(y) = 0$, for every y and $g(y) = |p_1(y)|$, then this gives a normed near-algebra, where as $f(y) = -p_2(y)$ and $g(y) = p_1(y)$ gives an algebra.

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