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MATRIX MAPS INVOLVING CERTAIN SEQUENCE SPACES

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This paper gives a characterization of BK-spaces which contain subspace isomorphic to $sc_0(\Delta)$ in terms of matrix maps and sufficient condition for a matrix map from $s\ell_\infty(\Delta)$ into a BK-space to be a compact operator. As a corollary, it is shown that any matrix map from $s\ell_\infty(\Delta)$ into a BK-space which does not contain any subspace isomorphic to $s\ell_\infty(\Delta)$ is compact.

§1. w denotes the space of all scalar sequences and any subspace of w is called sequence space.

The following sequence spaces will be used in the sequel:

ℓ_∞, the space of all bounded scalar sequences;
c_0, the space of all null scalar sequences;
ℓ_1, the space of all absolutely 1-summable scalar sequences;
Φ, the space of all finite scalar sequences;

$$s\ell_\infty(\Delta)=\{x=(x=(x_k)_{k=1}^\infty \in w: \Delta x=(x_k-x_{k+1})\in \ell_\infty, x_1=0\};$$
$$sc_0(\Delta)=\{x=(x_k)_{k=1}^\infty \in w: \Delta x=(x_k-x_{k+1})\in c_0, x_1=0\}. \qquad \ldots (1.1)$$

It is noted that $s\ell_\infty(\Delta)$ and $sc_0(\Delta)$ have been introduced by Kizmaz[8]. Besides these sequence spaces, all other sequence spaces mentioned above are well-known.

It is known that ℓ_∞ and c_0 with their usual norms, $\|x\|_\infty = \sup_k |x_k|$, are Banach spaces ($B$-spaces).

Following arguments given in Kizmaz[8] it can be easily proved that $s\ell_\infty(\Delta)$ and $sc_0(\Delta)$ with their norms,

$\|x\|_\Delta = \|\Delta x\|_\infty$, are B-spaces. It is also given in Kizmaz[8] that (1.1) $s\ell_\infty(\Delta)$ and ℓ_∞ are equivalent as topological spaces. We denote by e^k the kth unit vector. If E is a B-space, let E^* denotes its continuous dual and B_l denotes E's unit ball. The linear functional $p_k: w \to \mathbf{C}$ (complex numbers), defined by

$p_k(x)=x_k$, $x \in w$, is called the kth coordinate functional.

Definition 1—Let $(x^n)_{n=1}^{\infty} \subseteq E$, where E is a B-space. Then the series $\sum_{n=1}^{\infty} x^n$ is said to be unconditionally convergent in E if, given any permutation π of $\mathbf{N}$ (natural numbers), $\sum_{n=1}^{\infty} x^{\pi(n)}$ exists.

In B-space E, the following statements are equivalent[1,6]:

(i) $\sum_{n=1}^{\infty} x^n$ is unconditionally convergent.

(ii) $\sum_{n=1}^{\infty} x^n$ is weakly subseries convergent; that is, weak$_n$ lim $\sum_{j=1}^{n} x^{k_j}$ exists for each increasing sequence $(k_n)_{n=1}^{\infty}$ of positive integers.

(iii) $\sum_{n=1}^{\infty} x^n$ is subseries convergent; that is, norm$_n$ lim $\sum_{j=1}^{n} x^{k_j}$ exists with $(k_n)_{n=1}^{\infty}$ above.

(iv) $\sum_{n=1}^{\infty} x^n$ is bounded multiplier convergent; that is, $\sum_{n=1}^{\infty} x^n t_n$ exists for each sequence $t = (t_n)_{n=1}^{\infty}$ of bounded scalars. ... (1.2)

Definition 2—Let $(x^n)_{n=1}^{\infty} \subseteq E$, where E is a B-space. The series $\sum_{n=1}^{\infty} x^n$ is said to be weakly unconditionally Cauchy (wuC) if and only if for every $f \in E^*$, $\sum_{n=1}^{\infty} |f(x^n)| < \infty$; alternatively, $\sum_{n=1}^{\infty} x^n$ is wuC series if, given any permutation π of $\mathbf{N}$, $\left(\sum_{k=1}^{n} x^{\pi(k)}\right)$ is weakly Cauchy sequence.

We remark that a wuC series is not necessarily unconditionally convergent. For example, $\sum_{k=1}^{\infty} e^k$ is wuC series in c_0 but not unconditionally convergent in c_0. We refer Antosik and Swartz[1] and Diestel[6] for details.

We now define Banach coordinate spaces (BK-spaces) and matrix maps.

A BK-space E is a B-space which is a vector subspace of w together with $p_k \in E^*$ for every $k \in \mathbf{N}$. ℓ_∞, c_0, $s\ell_\infty(\Delta)$ and $sc_0(\Delta)$ with their norms mentioned earlier, are BK-spaces.

Let $A=(a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex numbers. Let E and F be BK-spaces. We write

$$Ax=(A_n(x))_{n\in \mathbf{N}} \text{ if } A_n(x)=\sum_{k=1}^{\infty} a_{nk}x_k \text{ converges for each } n\in \mathbf{N}.$$

If $x=(x_k)_{k=1}^{\infty}\in F$ then $Ax=(A_n(x))_{n\in\mathbf{N}}\in E$. We say that A defines a matrix map from F into E and we denote it by $A:F\to E$. It is proved in Theorem 7 (ii) of Maddox[9] that $A\in B(F,E)$, the space of all bounded (continuous) linear maps from F into E.

We denote by E_A the set
$E_A=\{x\in w: Ax \text{ exists and } Ax\in E\}$.

Remark: Note that A is a matrix map from F into E if and only if $F\subseteq E_A$.

We now suppose $A:F\to E$ defines a matrix map and let $A(e^k)=a^k$ denotes the kth column of the matrix A. It is observed[3,11] that if $\Phi\subseteq F$, elements of A are the elements of E. So there is no loss of generality if we discuss the convergence of the series $\sum_{k=1}^{\infty} a^k$.

From now on, E unless specified shall denote BK-space.

§2. We start with the following lemmas.

Lemma 1—Let $A:s\ell_\infty(\Delta)\to E$ defines a matrix map. If A is a weakly compact, then $\sum_{k=1}^{\infty} a^k$ is unconditionally convergent in E.

Proof: Let $t=(t_k)_{k=1}$ be an arbitrary sequence of scalares whose values are 0's and 1's. Then clearly $t\in s\ell_\infty(\Delta)$. Since $\left\{\sum_{k=1}^{m} t_k e^k: m\in \mathbf{N}\right\}$ is a bounded set in $s\ell_\infty(\Delta)$ and $A:s\ell_\infty(\Delta)\to E$ is a weakly compact map [that is, A maps every bounded set in $s\ell_\infty(\Delta)$ into a relatively weakly compact set in E (see Kantorovich and Akilov[7])], it follows that

$$K=\left\{A\sum_{k=1}^{m} t_k e^k: m\in \mathbf{N}\right\}=\left\{\sum_{k=1}^{m} t_k a^k: m\in \mathbf{N}\right\}$$

is a relatively weakly compact set in E. E is a BK-space, we have E's weak topology and the topology of coordinate-wise convergence inherited from w are equivalent on the weak of K (follows from the arguments that comparable Hausdorff topologies on a compact set are equal (see Bennet and Kalton[2]).

We now observe that $x = \sum_{k=1}^{\infty} t_k a^k$ is coordinatewise convergence if and only if $x_n = \lim_n \sum_{k=1}^{n} t_k a_{nk}$, $p_n(a^k) = a_{nk}$. Since $t \in s\ell_\infty(\Delta)$ and $A(t) = (A_n(t))_{n \in \mathbb{N}} \in E$, where

$A_n(t) = \lim_n \sum_{k=1}^{n} t_k a_{nk} = \sum_{k=1}^{\infty} t_k a_{nk}$; it follows that $A(t)$ is coordinatewise convergence, hence is the weak limit point of K. Accordingly, $\sum_{k=1}^{\infty} a^k$ is weakly subseries convergent in E; that is, $\sum_{k=1}^{\infty} a^k$ is unconditionally convergent, by (1.2).

Lemma 2—If $\sum_{k=1}^{\infty} a^k$ is unconditionally convergent in E, then

$A : s\ell_\infty(\Delta) \to E$ defines a matrix map, and

$$A(\alpha) = \sum_{k=1}^{\infty} a^k \alpha_k \text{ for every } \alpha = (\alpha_k)_{k=1}^{\infty} \in s\ell_\infty(\Delta).$$

PROOF: Since $\sum_{k=1}^{\infty} a^k$ is unconditionally convergent in E. Then from (1.2), $\sum_{k=1}^{\infty} a^k \alpha_k$ exists for every $\alpha = (\alpha_k)_{k=1}^{\infty} \in s\ell_\infty(\Delta)$. Also, we noted that

$$p_n\left(\sum_{k=1}^{\infty} a^k \alpha_k\right) = \sum_{k=1}^{\infty} a_{nk} \alpha_k = A_n(\alpha) = p_n(A(\alpha)), \text{ for every } n \in \mathbb{N}.$$ Since $(p_n)_{n=1}^{\infty}$ is total over E, it follows that

$$A(\alpha) = \sum_{k=1}^{\infty} a^k \alpha_k.$$

Now $\alpha = (\alpha_k)_{k=1}^{\infty} \in s\ell_\infty(\Delta)$ was chosen as arbitrary. So that $A(\alpha) \in E$ for every $\alpha \in s\ell_\infty(\Delta)$ and hence $A : s\ell_\infty(\Delta) \to X$ defines a matrix map.

Corollary 1—If $\sum_{k=1}^{\infty} a^k$ is unconditionally convergent in E, then $s\ell_\infty(\Delta) \subseteq E_A$.

Corollary 1 follows from Remark and Lemma 2.

Theorem 1—If $A : s\ell_\infty(\Delta) \to E$ is a weakly compact matrix map, then A is compact map.

PROOF : It is enough to prove that $A(B_{s\ell_\infty(\Delta)})$ is a relatively compact set in E; that is, for every $\varepsilon > 0$, there is a relatively compact set K_ε in E such that

$A(B_{s\ell_\infty(\Delta)}) \subseteq K_\varepsilon + \varepsilon B_E$.

We have, from Lemma 1, that

$\sum_{k=1}^{\infty} a^k$ is unconditionally convergent in E. So that for any $\alpha \in s\ell_\infty(\Delta)$ and for any $\delta > 0$, there is a N_δ such that

$$\left\| \sum_{k=m}^{\infty} a^k \alpha_k \right\| < \delta \, \|\alpha\|_\Delta \text{ for every } m \geq N_\delta. \quad \ldots (2.1)$$

We now suppose that

$A(B_{s\ell_\infty(\Delta)}) \subseteq NB_E$, $N > 0$.

Set $\varepsilon = \delta / N$. Choose M such that (2.1) is satisfied and let F_M be the linear span of $(a^k)_{k=1}^{M}$. Then $F_M \cap NB_E = K_\varepsilon$ is clearly a compact set in E, since F_M has finite dimension.

Finally, we have

$$A(B_{s\ell_\infty(\Delta)}) = \left\{ \sum_{k=1}^{\infty} a^k \alpha_k : \alpha \in B_{s\ell_\infty(\Delta)} \right\} \subseteq K_\varepsilon + \delta \frac{B_E}{N} = K_\varepsilon + \varepsilon B_E .$$

Rosenthal theorem[10] states that any continuous linear map from ℓ_∞ into a B-space that does not contain any subspace isomorphic to ℓ_∞ is weakly compact.

Corollary 2—Let E be a BK-space such that it contains no subspace isomorphic to $s\ell_\infty(\Delta)$. If $A : s\ell_\infty(\Delta) \to E$ defines a matrix map, then A is compact map.

This follows from (1.1), H.P. Rosenthal theorem and Theorem 1.

§3. In this section we give characterization of BK-spaces that contain a subspace isomorphic to $sc_0(\Delta)$ in terms of matrix maps.

In order to prove the result we need the following lemmas.

Lemma 3—Let $(x^n)_{n=1}^{\infty} \subseteq E$, where E is a B-space. The following statements are equivalent.

(i) $\sum_{n=1}^{\infty} x^n$ is wuC.

(ii) There is $M > 0$ such that, for any $(\alpha_n)_{n=1}^{\infty} \in s\ell_\infty(\Delta)$, $\sup_n \left\| \sum_{k=1}^{n} \alpha_k x^k \right\| \leq M \|\alpha\|_\Delta$

(iii) For any $(\alpha_n)_{n=1}^{\infty} \in sc_0(\Delta)$, $\sum_{n=1}^{\infty} \alpha_n x^n$ converges.

(iv) There is $M > 0$ such that for any finite set $\sigma \subseteq \mathbb{N}$ and any *sgns* $\pm$, we have

$\left\| \sum_{n \in \sigma} \pm x^n \right\| \leq M$.

PROOF : (i) ⇒ (ii):

Define $S: E^* \to \ell_1$ by

$$S(f) = (f(x^n)),\ f \in E^*.$$

Then it is easy to see that S is a well-defined linear map with a closed graph. So, S is continuous (bounded) and that for any $(a_n)_{n=1}^{\infty} \in B_{sc_0(\Delta)}$ and for any $f \in B_{E^*}$, we have

$$\left| f\left(\sum_{k=1}^{n} a_k x^k\right) \right| = |\langle (0, a_2, a_3, \ldots, a_n, 0, 0, \ldots), S(f) \rangle| \leq \|S\|, \text{ since } \|f\| \leq 1$$

and $\|a\|_\Delta \leq 1$.

Hence (i) ⇒ (ii).

(ii) ⇒ (iii).

Let $(a_n)_{n=1}^{\infty} \in sc_0(\Delta)$. Keeping $m < n$ and allowing $m, n \to \infty$, we have

$$\left\| \sum_{k=m}^{n} a_k x^k \right\| \leq M \sup_{m \leq k < n} |a_k - a_{k+1}| \to 0.$$

Hence, (ii) ⇒ (iii).

(iii) ⇒ (iv):

Define $T_\Delta : sc_0(\Delta) \to E$ by

$$T_\Delta(a_n) = \sum_{n=1}^{\infty} a_n x^n,\ (a_n)_{n=1}^{\infty} \in sc_0(\Delta).$$

Clearly T_Δ is bounded. So values of T_Δ on $B_{sc_0(\Delta)}$ are also bounded. Hence, there is a $M > 0$ such that for any finite set $\sigma \subseteq \mathbf{N}$ and any *sgns* ±, we have

$$\left\| \sum_{n \in \sigma} \pm x^n \right\| \leq M.$$

Thus, (iii) ⇒ (iv).

(iv) ⇒ (i): For any $f \in B_{E^*}$, we have

$$f\left(\sum_{n \in \sigma} \pm x^n\right) = \sum_{n \in \sigma} \pm f(x^n) \leq \left\| \sum_{n \in \sigma} \pm x^n \right\| \leq M$$

for any $\sigma \subseteq \mathbf{N}$ and any *sgns* ±, since (iv) holds.

Hence, (iv) ⇒ (i).

This completes the proof.

Lemma 4—$A : sc_0(\Delta) \to E$ defines a matrix map if and only if $\sum_{k=1}^{\infty} a^k$ is *wuC* series in E.

PROOF : Necessity: Following arguments given in Maddox[9] it can be easily shown that $\{e^k\}_{k=1}^{\infty}$ is a Schauder basis for $sc_0(\Delta)$. So that for any $a = \{a_k\}_{k=1}^{\infty} \in sc_0(\Delta)$, we have

$$a = \lim_n \sum_{k=1}^{n} a_k e^k$$

By the linearity and continuity of A,

$$A(a) = \lim_n \sum_{k=1}^{n} a_k A(e^k) = \sum_{k=1}^{\infty} a_k a^k.$$

Moreover, $A : sc_0(\Delta) \to E$ defines a matrix map. It follows that $\sum_{k=1}^{\infty} a^k a_k$ exists for every $a \in sc_0(\Delta)$. Hence by Lemma 3, $\sum_{k=1}^{\infty} a^k$ is wuC series in E.

Sufficiency: Suppose that $\sum_{k=1}^{\infty} a^k$ is wuC series in E. Then by Lemma 3, $\sum_{k=1}^{\infty} a^k a_k$ is convergent for every $a = (a_k)_{k=1}^{\infty} \in sc_0(\Delta)$. Now an arrangement similar to the one given in Lemma 2 shows that

$$A(a) = \sum_{k=1}^{\infty} a^k a_k \text{ for every } a \in sc_0(\Delta).$$

Hence, $sc_0(\Delta) \subseteq E_A$ and $A : sc_0(\Delta) \to E$ defines a matrix map. This completes the proof.

We also use the following theorem.

Theorem (*Bessaga-Pelczynaski*)—Let E be a B-space. E contains a subspace isomorphic to c_0 if and only if E contains wuC series which is not unconditionally convergent[1,0]. This, in consequence with $sc_0(\Delta) \sim c_0$ (see Kizmaz[6]) and Lemma 4, gives the following corollary.

Corollary 3—Let E be a BK-space such that E contains a sub-space isomorphic to $sc_0(\Delta)$. If A be a matrix such that $sc_0(\Delta) \subseteq E_A$, then $\sum_{k=1}^{\infty} a^k$ is wuC series in in E which is not unconditionally convergent.

We now prove the following theorem.

Theorem 2—E does not contain any subspace isomorphic to $sc_0(\Delta)$ if and only if whenever A be a matrix such that $sc_0(\Delta) \subseteq E_A$, then $s\ell_\infty(\Delta) \subseteq E_A$ and $A : s\ell_\infty(\Delta) \to E$ defines a compact map.

PROOF : Necessity: Suppose E contains no subspace isomorphic to $sc_0(\Delta)$ and

A is a matrix such that $sc_0(\Delta) \subseteq E_A$. Lemma 4 shows that $\sum_{k=1}^{\infty} a^k$ wuC series in E and therefore, from Bessaga-Pelczynaski Theorem, that $\sum_{k=1}^{\infty} a^k$ is unconditionally convergent in E. Consequently, from Corollary 1 and the proof of Theorem 1, it follows that $s\ell_\infty(\Delta) \subseteq E_A$ and $A : s\ell_\infty(\Delta) \to E$ is a compact map.

Sufficiency: If $\sum_{k=1}^{\infty} a^k$ is wuC series in E, then the matrix A given by $a_{nk} = p_n(a^k)$ for all $n, k \in \mathbf{N}$ has the property that $sc_0(\Delta) \subseteq E_A$. Consequently, $s\ell_\infty(\Delta) \subseteq E_A$ and $A : s\ell_\infty(\Delta) \to E$ is a compact map, which implies from Lemma 1 that $\sum_{k=1}^{\infty} a^k$ is unconditionally convergent in E and hence from Bessage-Pelczynaski Theorem E cannot contain any subspace isomorphic to $sc_0(\Delta)$.

Corollary 4—Let E be a BK-space such that it contains no subspace isomorphic to $s\ell_\infty(\Delta)$ (that is, E is separable). E contains no subspace isomorphic to $sc_0(\Delta)$ if and only if, whenever A be a matrix such that $sc_0(\Delta) \subseteq E_A$, then $s\ell_\infty(\Delta) \subseteq E_A$.

This follows from Theorem 2 and Corollary 2.

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