Indian J. pure appl. Math., 24(3): 145-149, March 1993

ON SOME NONEXPANSIVE TYPE MAPPINGS AND FIXED POINTS

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(*Received* 11 *August* 1992; *accepted* 4 *December* 1992)

In this paper a class of selfmappings of metric spaces, which satisfy the contractive definition (4) below, are considered and a fixed point theorem is established.

1. INTRODUCTION

Let (X, D) be a metric space and T a mapping of X into itself. If there exists a nonnegative real number k such that $k < 1$ (resp. $k = 1$) and for any x, $y \in X$,

$$d(Tx, Ty) \leq kd(x, y) \quad \ldots (1)$$

then T is said to be a contraction (resp. nonexpansive) mapping. Banach contraction mapping principle states that if X is complete, then every contraction mapping has a unique fixed point and this point can be obtained by repeated iteration of the mapping. However nonexpansive mapping may be without fixed points.

Generalizing contraction mappings Ćirić[2] introduced and investigated mappings (not necessarily continuous) which satisfy the following contractive condition:

$$d(Tx, Ty) \leq a_1 d(x,y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4[d(x, Ty) + d(y, Tx)] \quad \ldots (2)$$

for all $x, y \in X$, where $a_i = a_i(x,y)$ $(i = 1, 2, 3, 4)$ are nonnegative real function such that

$$(a_1 + a_2 + a_3 + 2a_4)(x, y) \leq a < 1 \quad \ldots (3)$$

where a is a constant real number. The class of contractions, were further extended and investigated by Pal and Maiti[5], Ray[6], Reily and Subrahmanum[7], Rhoades[4] and others.

This purpose of this paper is to investigate a class of selfmappings T which satisfy the following contractive definition:

$$d(Tx, Ty) \leq a \cdot \max\{d(x,y), d(x, Tx), d(y, Ty), 1/2 \cdot [d(x, Ty) + d(y, Tx)]\} + b \cdot \max\{d(x, Tx), d(y, Ty)\} + c[d(x, Ty) + d(y, Tx)] \quad \ldots (4)$$

for all $x, y \in X$, where a, b, c are nonnegative real numbers such that

$$a + b + 2c = 1. \qquad \ldots (5)$$

Note that (2) and (3) imply (4) and (5) with $b = c = (1 - a)/2$.

2. Main Results

We begin by analyzing the properties of a mapping T which satisfies (4) with $a, b, c \geq 0$.

Proposition 2.1—Let $T: X \to X$ be a mapping satisfying (4), with $a + 2c < 1$. Then T has at most one fixed point.

Proof: If u and v are fixed points of T, then by (4),

$$d(u, v) = d(Tu, Tv) \leq ad(u, v) + 2cd(u, v).$$

Hence

$$0 \leq (1 - a - 2c)\, d(u, v) \leq 0$$

implying that $u = v$.

Proposition 2.2—Let $X \to X$ be a mapping satisfying (4), where a, b, c satisfy (5) and $b > 0$, $c > 0$. Then

$$\inf\{d(x, Tx) : x \in X\} = 0. \qquad \ldots (6)$$

Proof: Let x_0 be an arbitrary point in X and define the Picard iterates $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \cdots$. Then by (4)

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) = d(Tx_n, Tx_{n+1}) \leq a \cdot \max\{&d(x_n, x_{n+1}), \\ &d(x_{n+1}, Tx_{n+1}), 1/2 \cdot d(x_n, Tx_{n+1})\} \\ &+ b \cdot \max\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &+ c \cdot d(x_n, Tx_{n+1}). \end{aligned}$$

Hence, applying the triangle inequality, we get

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) \leq (a + b) \max\{&d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\ &+ c[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})]. \end{aligned}$$

If $d(x_{n+1}, Tx_{n+1}) > d(x_n, Tx_n)$ for some n, then we have,

$$\begin{aligned} d(x_{n+1}, Tx_{n+1}) &< (a + b)\, d(x_{n+1}, Tx_{n+1}) + 2cd(x_{n+1}, Tx_{n+1}) \\ &= d(x_{n+1}, Tx_{n+1}) \end{aligned}$$

a contradiction. Thus $d(x_{n+1}, Tx_{n+1}) \leq d(x_n, Tx_n)$.

Hence

$$d(x_n, Tx_n) \leq d(x_0, Tx_0) \qquad \ldots (7)$$

for all positive integers n.

Using (4) again, by (7) we have

$$d(x_1, Tx_2) = d(Tx_0, Tx_2) \le a \max\{d(x_0, Tx_1), d(x_0, Tx_0), 1/2[d(x_0, Tx_2) + d(x_0, Tx_0)]\} + b\, d(x_0, Tx_0) + c[d(x_0, Tx_2) + d(x_0, Tx_0)]. \quad \ldots (8)$$

Since by (7) and the triangle inequality

$$1/2[d(x_0, Tx_2) + d(x_0, Tx_0)] \le 1/2[d(x_0, Tx_1) + d(x_1, Tx_1) + d(x_2, Tx_2) + d(x_0, Tx_0)] \le 2\,d(x_0, Tx_0), d(x_0, Tx_2) + d(x_0, Tx_0) \le 2\,d(x_0, Tx_0) + d(x_1, Tx_2)$$

from (8) and (5) we have

$$d(x_1, Tx_2) \le 2ad(x_0, Tx_0) + bd(x_0, Tx_0) + 2cd(x_0, Tx_0) + cd(x_1, Tx_2) = (1 + a)d(x_0, Tx_0) + cd(x_1, Tx_2).$$

Hence

$$d(x_1, Tx_2) \le (1 + a)/(1 - c) \cdot d(x_0, Tx_0) \le (2 - b)d(x_0, Tx_0). \quad \ldots (9)$$

Using (4), (7) and (9) we have

$$d(x_2, Tx_2) = d(Tx_1, Tx_2) \le a \max\{d(x_0, Tx_0), 1/2\, d(x_1, Tx_2)\} + bd(x_0, Tx_0) + cd(x_0, Tx_2)] \le ad(x_0, Tx_0) + bd(x_0, Tx_0)] + c(2 - b)d(x_0, Tx_0) \quad \ldots (10)$$

and hence

$$d(x_2, Tx_2) \le (1 - bc)d(x_0, Tx_0).$$

It is easily shown by induction that (10) implies

$$d(x_n, Tx_n) \le (1 - bc)^{(n/2)} d(x_0, Tx_0) \quad \ldots (11)$$

where $(n/2)$ means the greatest integer not exceeding $n/2$. Since by hypothesis $bc > 0$, from (11) we conclude that (6) holds. The proof is complete.

Let X be a metric space, T be a mapping of X into itself, and x be a point in X. The mapping T is said to be 'asymptotically' regular at x if

$$\lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0.$$

Because of the definition of x_n in the proof of Proposition 2.2 we observe that by (11)

$$d(T^n x_0, T^{n+1} x_0) = d(x_n, Tx_n) \to 0 \text{ as } n \to \infty$$

so that we have

Proposition 2.3—Let $T: X \to X$ be a mapping satisfying (4), where $a \ge 0$, $b > 0$, $c > 0$ and (5) holds. Then T is asymptotically regular at any point in X.

Note that, if T satisfies (4), where a, b, c satisfy (5), then T may be discontinuous at some points. But we have the following result.

Proposition 2.4—If T satisfies (4), where $a \geq 0$, $b \geq 0$, $c > 0$ and (5) holds, and if T has a fixed point p, then T is continuous at p.

PROOF: Let $x_n \to p = Tp$. Then from (4),

$$\begin{aligned} d(Tx_n, Tp) &\leq a \max\{d(x_n, p), d(x_n, Tx_n), 1/2[d(x_n, p) + d(Tx_n, Tp)]\} \\ &\quad + bd(x_n, Tx_n) + c[d(x_n, p) + d(Tx_n, Tp)] \\ &\leq a[d(x_n, p) + d(Tp, Tx_n)] + (b + c)[d(x_n, p) + d(Tx_n, Tp)] \end{aligned}$$

and hence, as $a + b + c = 1 - c < 1$, we have

$$d(Tx_n, Tp) \leq (1 - c)/c \cdot d(x_n, p).$$

Hence it follows that $Tx_n \to Tp$. Therefore, T is continous at p.

Using Propositions 2.1, 2.2 and 2.4 we have the following fixed point theorem.

Theorem 2.1—Let (X, d) be a complete metric space and $T: X \to X$ a mapping satisfying (4), where $a \geq 0$, $b > 0$, $c > 0$ and such that (5) holds. Then T has a unique fixed point, say $p \in X$, and at this point T is continous. Moreover, for any x in X,

$$d(T^n x, p) \leq 2/bc \cdot (1 - bc)^n \, d(x, Tx). \qquad \ldots (12)$$

PROOF: Let $x = x_0 \in X$ be arbitrary and define the Pickard iterates $x_{n+1} = Tx_n$, for $n = 0, 1, 2, \ldots$. Then, by (11), we conclude that $\{T^n x\}$ is a Caushy sequence. Since X is complete, there is some $p \in X$ such that

$$\lim_{n \to \infty} T^n x = p.$$

Using (4) we have

$$\begin{aligned} &d(T^n x, Tp) a \max\{d(T^{n-1}x, p), d(T^{n-1}x, T^n x), d(p, Tp), \\ &1/2[d(T^{n-1}x, Tp) + d(p, T^n x)]\} + b \max\{d(T^{n-1}x, T^n x), d(p, Tp)\} \\ &+ c[d(T^{n-1}x, Tp) + d(p, T^n x)]. \end{aligned}$$

Taking the limit as $n \to \infty$ yields

$$d(p, Tp) \leq ad(p, Tp) + bd(p, Tp) + cd(p, Tp) = (1 - c)\, d(p, Tp)$$

which implies $d(p, Tp) = 0$, as $c > 0$. Thus $Tp = p$. By Propositions 2.1 and 2.4, p is a unique fixed point and at this point T is continuous. By (11) it is easy to obtain (12). The proof is complete.

Corollary 2.1 (Bogin[1]) — Let (X, d) be a complete metric space and $T: X \to X$ a mapping satisfying

$$\begin{aligned} d(Tx, Ty) &\leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] \\ &\quad + c[d(x, Ty) + d(y, Ty)] \qquad \ldots (13) \end{aligned}$$

where $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

We point out that Theorem 2.1 generalizes the main result of Ćirić[2], Hardy and Rogers[3] and a great number of known generalizations of Banach contraction principle (see Rhoades[4]). The following example shows that Theorem 2.1 is a stronger result then Theorem of Bogin[1] and a coresponding theorem of Hardy and Rogers[3].

Example—Let $X = \{1, 3, 4, 6\}$ be a set of positive integers with usual distance. Define

$$T1 = 6,\ T6 = 4,\ T4 = T3 = 3.$$

Then it is easy to see that T satisfies (4) with, for example, $a = 3.5$, $b = 1.5$, $c = 1/10$. Since X is complete, we can apply Theorem 2.1. To see that T does not satisy (13, let $x = 1$ and $y = 3$. Then we have

$$ad(x,y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] = 2a + 5b + 5c$$
$$\leq (a + 2b + 2c) \cdot 5/2 = 1.5/2 < 3 = d(Tx, Ty)$$

for any a, b, $c \geq 0$ with $a + 2b + 2c = 1$.

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