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FIXED POINT THEOREMS FOR NON-SELF MAPS III

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Let X be a complete metric space. Fixed point theorems are proved for non-self maps. Also, several examples are given involving self maps and Caristi type conditions.

Suppose (X, d) is a complete metric space, $T: X \to X$, and ϕ is lower semi-continuous where $\phi: X \to [0, \infty)$. Consider the following conditions.

(1) $d(x, Tx) \leqslant \phi(x) - \phi(Tx)$ for every x in X.

(2) $d(x, Tx) \geqslant \phi(Tx) - \phi(x)$ for every x in X.

(3) $d(x, Tx) \geqslant \phi(x) - \phi(Tx)$ for every x in X.

(4) $d(x, Tx) \leqslant \phi(Tx) - \phi(x)$ for every x in X.

Caristi's theorem[1] says that if (1) holds, then T has a fixed point in X. Park[2] proved that if T is onto and (4) holds, then T has a fixed point. We give an example to show that T is onto is necessary for Park's theorem. Also, examples are given to show that conditions (2) and (3) need not give a fixed point. Then a general theorem is proved that yields several results even for certain non-self maps.

Example 1—We show that T is onto is necessary for Park's theorem[2]. Let X be the reals and set $Tx = 2x - 1$ for $x \leqslant 0$ and $Tx = 2x + 1$ for $x > 0$. Then $d(Tx, Ty) \geqslant 2\, d(x, y)$. Consider $\phi(x) = d(x, Tx)$. Now

$$\lim_{x \to 0^-} d(x, Tx) = \lim_{x \to 0} d(x, Tx) = 1 = d(0, T(0))$$

gives ϕ is a continuous function. T is not onto and T is fixed point free. We show that (4) holds. $d(x, Tx) \leqslant \frac{1}{2}\, d(Tx, T^2x)$ so

$$-\frac{1}{2}\, d(x, Tx) + d(x, Tx) \leqslant -\frac{1}{2}\, d(x, Tx) + \frac{1}{2}\, d(Tx, T^2x),$$

$$\text{or } d(x, Tx) \leqslant [d(Tx, T^2x) - d(x, Tx)] = \phi(Tx) - \phi(x).$$

Example 2—Suppose $d(Tx, Ty) \leqslant k\,d(x, y)$ where $k > 1$. Let $y = Tx$, add $-d(x, Tx)$ to both sides and solve for $d(x, Tx)$. Then

$$d(x, Tx) \geqslant \frac{1}{k-1}[d(Tx, T^2x) - d(x, Tx)]$$

or $d(x, Tx) \geqslant \phi(Tx) - \phi(x)$, if we let $\phi(x) = \frac{1}{k-1}\, d(x, Tx)$.

Now let $X = [1, \infty)$ and $Tx = x + \frac{1}{x}$. Then $|Tx - Ty| < |x - y| < 2|x - y|$ for $x \neq y$. Thus, T and ϕ are both continuous, T is fixed point free, and (2) holds.

Example 3—Let X denote the real numbers and set $Tx = x + 1$. Let $0 < k < 1$, and set $\phi(x) = \frac{1}{1-k}\, d(x, Tx)$. T does not have a fixed point and ϕ is continuous. We show that (3) holds.

$$d(Tx, Ty) = d(x, y) > k\,d(x, y) \text{ and } k\,d(x, Tx) < d(Tx, T^2x).$$

Therefore,

$$-d(x, Tx) + k\,d(x, Tx) < -d(x, Tx) + d(Tx, T^2x),$$

$$(1-k)\,d(x, Tx) > d(x, Tx) - d(Tx, T^2x),$$

$$d(x, Tx) > \frac{1}{1-k}[d(x, Tx) - d(Tx, T^2x)],$$

or $d(x, Tx) > \phi(x) - \phi(Tx)$ which is (3).

Theorem (Ekeland[2,3])—Every lower semi-continuous function ϕ from a complete metric space X into $[0, \infty)$ has a d-point p in X; that is, for every $x \neq p$ we have $\phi(p) - \phi(x) < d(p, x)$.

Theorem 1—Let C be a closed subset of a metric space X, $T: C \to X$ with $C \subset T(C)$, and $g: C \to C$. Suppose $\phi: C \to [0, \infty)$ is lower semi-continuous and $d(Tx, gx) \leqslant \phi(Tx) - \phi(gx)$ for every x in $T^{-1}(C)$. Then $T(x_0) = g(x_0)$ for some x_0 in C.

Proof: From Ekeland's theorem there exists p in C with $\phi(p) - \phi(x) < d(x, p)$ for all x in C with $x \neq p$. $p \in C \subset T(C)$ so $p = T(x_0)$ for some $x_0 \in C$. Now if $T(x_0) \neq g(x_0)$, $T(x_0) = p \in C$ and $g(x_0) \in C$ so that $\phi(Tx_0) - \phi(gx_0) < d(Tx_0, gx_0)$, a contradiction.

Corollary 1—Suppose C is a closed subset of the complete metric space X and $T: C \to X$ with $C \subset T(C)$. Suppose $\phi: C \to [0, \infty)$ is lower semi-continuous and for every $x \in T^{-1}(C)$, $d(Tx, x) \leqslant \phi(Tx) - \phi(x)$. Then $Tx_0 = x_0$ for some x_0 in C.

Proof: Let $g = I$ in Theorem 1.

Corollary 2—Suppose C is a closed subset of a complete metric space X and $T: C \to X$ is continuous with $C \subset T(C)$. If there exists $k > 1$ such that for every x in $T^{-1}(C)$, $d(Tx, T^2x) \geq k\, d(x, Tx)$, then T has a fixed point in C.

PROOF: Set $\alpha = \frac{1}{k}$ and $\phi(x) = \frac{\alpha}{1-\alpha}\, d(x, Tx)$. $d(x, Tx) \leq \alpha\, d(Tx, T^2x)$ gives

$$-\alpha\, d(x, Tx) + d(x, Tx) \leq \alpha\, d(Tx, T^2x) - \alpha\, d(x, Tx),$$

$$d(x, Tx) \leq \frac{\alpha}{1-\alpha}\, [d(Tx, T^2x) - d(x, Tx)],$$

or $d(x, Tx) \leq \phi(Tx) - \phi(x)$ and ϕ is continuous. Apply Corollary 1.

Suppose $0 < k < 1$ and $T: X \to X$ satisfies $d(Tx, T^2x) \leq k\, d(x, Tx)$ for each x. Hicks and Rhoades[4] and Rhoades[10] have shown that a general fixed point theorem results and many theorems, involving rather complicated conditions, are corollaries of this theorem. For example, $(21)'$ reduces to the above condition where $(21)'$ is as follows. For

$$0 < h < 1,\ d(Tx, Ty) \leq h \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}$$

The number $(21)'$ comes from an important paper by Rhoades[7], where he partially ordered most known contractive conditions. One can think of the condition in Corollary 2 as the dual of $d(Tx, T^2x) \leq k\, d(x, Tx)$. We show that Corollary 2 has many corollaries. If we replace x by Tx, y by Ty, Tx by x, and Ty by y in $(21)'$ we obtain $(21)^*$; that is, for $0 < h < 1$,

$$d(x, y) \leq h \max \left\{ d(Tx, Ty), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}$$

Remarks: Suppose C is a closed subset of a complete metric space X. $T: C \to X$ is continuous with $C \subset T(C)$, and $(21)^*$ holds. Then T has a fixed point in C. For, if $x \in T^{-1}(C)$, set $y = Tx$ in $(21)^*$. Then it follows that $d(Tx, T^2x) \geq k\, d(x, Tx)$ where $k > 1$. Thus Corollary 2 applies.

One can formulate definitions $(1)^*$, $(4)^*$, $(5)^*$, $(7)^*$, $(9)^*$, $(11)^*$, $(18)^*$, and $(19)^*$ from (1), (4), etc. of Rhoades[9] analogous to $(21)^*$ above. Each of these are special cases of $(21)^*$. Hence, all are special cases of Corollary 2, or Corollary 1, if one removes the hypothesis of continuity of T, and adds lower semi-continuity of $d(x, Tx)$.

Fixed point theorems for non-self maps are unusual. It is well known that a continuous function from an arc onto a containing arc must have a fixed point. Therefore, the condition $C \subset T(C)$ is a reasonable one. In[7], Sam Nadler showed

the following. For $n \geq 2$, let A and B be closed balls in E^n where $A \subset B$ and $A \neq B$. There exists functions f and g such that:

(a) f maps A onto B, $f(\partial A) = B$, f is continuous, and f is fixed point free,

(b) g maps A onto B, $g^{-1}(\partial B) = \partial A$, g is continuous, and g is fixed point free.

Other results for non-self maps can be found in Hicks and Saliga[5,6].

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