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COMPARATIVE GROWTH OF COMPOSITION OF ENTIRE FUNCTIONS

ANAND P. SINGH* AND M. S. BALORIA
Department of Mathematics, University of Jammu, Jammu 180 004

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Relative growth of one set of composite entire functions have been compared with another set of composite entire functions.

INTRODUCTION

In a recent paper[5] the authors proved that if f and g are entire functions of positive lower order and of finite order then

$$\lim_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(r^A, f)} = \infty$$

for every positive constant A. A question one can ask is whether the following is also true? If f, g, h are entire functions of positive lower order and of finite order then

$$\lim_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(r^A, f\circ h)} = \infty \qquad \ldots (1)$$

for every positive constant A? The answer to the above question is clearly no. For, let f be any entire function of positive lower order and of finite order and let $f = g = h$ and $A = 1$. Then the limit in (1) equals 1. However, if we put some restrictions on the growths of g and h and on the constant A then the statement (1) would be true. Observe that in (1) we have compared the growths of two entire functions having the same left factor. In this paper we shall also be comparing the growths of two entire functions having the same right factor and also with entire functions having different left and right factors.

STATEMENT OF THE MAIN RESULTS

Throughout this paper ρ_f and λ_f will respectively denote order and lower order of the entire function f.

*Present address: Department of Mathematics, Imperial College of Science and Technology, Huxley Building, 180 Queen's Gate, London SW7 2BZ, England.

Theorem 1—Let f, g, h be entire functions with positive lower order and of finite order. Further let $\rho_h < \lambda_g$. Then

$$\lim_{r\to\infty} \frac{\log\log M(r, f\circ g)}{\log\log M(r^A, f\circ h)} = \infty$$

for every positive constant A satisfying $A\rho_h < \lambda_g$.

Note: The restriction on the constant A is essential. For let $f(z) = h(z) = e^z$ and $g(z) = e^{z^2}$ then $\lambda_g = \rho_g = 2$, $\rho_h = 1$. Let $A = 3$ then $A\rho_h \not< \lambda_g$ and clearly

$$\frac{\log\log M(r, f\circ g)}{\log\log M(r^3, f\circ h)} = \frac{r^2}{r^3} \to 0 \text{ as } r\to\infty.$$

For composite entire functions having the same right factor we have the following theorem.

Theorem 2—Let f, g, h be entire functions of finite order. Let $\lambda_h > 0$, $\rho_f < \rho_h$, then

$$\liminf_{r\to\infty} \frac{\log\log M(r/2, f\circ g)}{\log\log M(r, h\circ g)} \leqslant \frac{\rho_h}{\lambda_h}.$$

If f and h both have positive lower order and $T(r,f)$ denotes the Nevanlinna characteristic function of f, then we have the following theorem.

Theorem 3—Let f, g and n be transcendental entire functions of finite order with $\lambda_h > 0$. Then

$$\liminf_{r\to\infty} \frac{\log T(r, f\circ g)}{\log T(r, h\circ g)} \leqslant \frac{\rho_f}{\lambda_h}.$$

In our next theorem we show that if $\lambda_h > 0$ and order of k is even 'slightly' larger than order of g then also the Nevanlinna characteristic of $h\circ k$ grows at a very fast rate in comparison to the Nevanlinna characteristic of $f\circ g$. Thus we shall prove the following theorem.

Theorem 4—Let g, h, k be transcendental entire functions of finite order. Further let $\lambda_h > 0$ and $0 < \rho_g < \rho_k$. Then for every transcendental entire function f of finite order

$$\limsup_{r\to\infty} \frac{T(r, h\circ k)}{T(r, f\circ g)} = \infty.$$

If further $\lambda_k > \rho_g$, we can improve Theorem 4 and prove that the limit of above ratio is infinite. Thus we shall prove:

Theorem 5—Let g, h, k be transcendental entire functions of finite order. Further let $\lambda_h > 0$ and $0 < \rho_g < \lambda_k$ then for every transcendental entire function f of finite order,

$$\lim_{r\to\infty} \frac{T(r, h\circ k)}{T(r, f\circ g)} = \infty.$$

For composite entire functions having right factors of regular growth (i.e. entire function having the same order as well as lower order) we have the following result.

Theorem 6—Let f, g, h, k be transcendental entire functions of finite order and of positive lower order. Further let $\lambda_g = \rho_g$ and $\lambda_k = \rho_k$. Then

$$\lim_{r \to \infty} \frac{\log \log T(r, f \circ g)}{\log \log T(r, h \circ k)} = \frac{\rho_g}{\rho_k}.$$

Preliminaries

For the proof of the theorems we shall need the following result of Clunie[1].

Theorem A—Let f and g be entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = (1 - \alpha)^2/4\alpha$. Then for $r > 0$,

$$M(r, f \circ g) \geq M(c(\alpha) M(\alpha r, g), f). \quad \ldots (2)$$

The opposite inequality

$$M(r, f \circ g) \leq M(M(r, g), f) \quad \ldots (3)$$

is an immediate consequence of the definition.

Another result that we shall need is the following relation between $T(r, f)$ and $M(r, f)$ of Hayman[2].

Theorem B—If f is analytic for $|z| \leq R$, then for $0 \leq r < R$,

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R + r}{R - r} T(R, f).$$

Theorem C (Niino and Suita[3])—Let $f(z)$ and $g(z)$ be entire functions. Let $g(0) = 0$. Then

$$T(r, f \circ g) \leq T(M(r, g), f)$$

for all $r > 0$.

Proof of Theorems

Proof of Theorem 1—Putting $\alpha = 1/2$ in (2) and taking r sufficiently large it follows that

$$M(r, f \circ g) \geq M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right)$$

$$\geq M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right).$$

Thus for all $r \geqslant r_0$,

$$\log\log M(r, f\circ g) \geqslant \log\log M\left(\frac{1}{16}M\left(\frac{r}{2}, g\right), f\right)$$
$$> (\lambda_f - \varepsilon)\log\left(\frac{1}{16}M\left(\frac{r}{2}, g\right)\right)$$
$$= (\lambda_f - \varepsilon)\log\frac{1}{16} + (\lambda_f - \varepsilon)\log M\left(\frac{r}{2}, g\right)$$
$$> (\lambda_f - \varepsilon)\log\frac{1}{16} + (\lambda_f - \varepsilon)\left(\frac{r}{2}\right)^{\lambda_g - \varepsilon} \qquad \ldots(4)$$

On the other hand using the inequality (3), we have,

$$\log\log M(r, f\circ h) \leqslant \log\log M(M(r,h), f)$$
$$< (\rho_f + \varepsilon)\log M(r,h), \text{ for all } r \geqslant r_0^*$$
$$< (\rho_f + \varepsilon) r^{\rho_h + \varepsilon}, \text{ for all } r \geqslant r_0.$$

Thus for all $r \geqslant r_0$,

$$\frac{\log\log M(r, f\circ g)}{\log\log M(r^A, f\circ h)} > \frac{(\lambda_f - \varepsilon)\log\frac{1}{16} + (\lambda_f - \varepsilon)\left(\frac{r}{2}\right)^{\lambda_g - \varepsilon}}{(\rho_f + \varepsilon)(r^A)^{\rho_h + \varepsilon}}$$
$$\to \infty \text{ as } r \to \infty,$$

since we can choose $\varepsilon > 0$ such that $\lambda_g - \varepsilon > A(\rho_h + \varepsilon)$.

Proof of Theorem 2—Since $\rho_f < \rho_\varphi$, there exists $\alpha > 0$ such that $\rho_f + \alpha = \rho_h$. Next, by definition of order of f, for the above $\alpha > 0$, there exists a positive number r_0 such that for all $r \geqslant r_0$

$$\frac{\log\log M(r, f)}{\log r} < \rho_f + \alpha = \rho_h.$$

And so for all $r \geqslant r_0$

$$\log M(r, f) < r^{\rho_h}. \qquad \ldots(5)$$

Thus

$$\log\log M(r, f\circ g) \leqslant \log\log M(M(r,g), f)$$
$$< \log\{M(r,g)\}^{\rho_h}$$
$$= \rho_h \log M(r,g) \qquad \ldots(6)$$

for all $r \geqslant r_0$.

*r_0 need not be the same at every stage.

Next, as in Theorem 1, for all $r \geq r_0$

$$\log \log M(r, h \circ g) > A + (\lambda_h - \varepsilon) \log M\left(\frac{r}{2}, g\right)$$

where $A = (\lambda_h - \varepsilon) \log \frac{1}{16}$.

Now let $\lambda_g(r)$ be a lower proximate order relative to $\log M(r, g)$ and $p > 0$ be any constant. Then

$$\log \log M(r, h \circ g) > A + (\lambda_h - \varepsilon)\left(\frac{r}{2}\right)^{\lambda_g(r/2)}, \text{ for all } r \geq r_0$$

$$\sim (\lambda_h - \varepsilon)\left(\frac{1}{2}\right)^{\lambda_g} r^{\lambda_g}, \text{ as } r \to \infty$$

$$\sim (\lambda_h - \varepsilon)\left(\frac{1}{2p}\right) p^{\lambda_g} r^{\lambda_g}$$

$$\sim (\lambda_h - \varepsilon)\left(\frac{1}{2p}\right)^{\lambda_g} (pr)^{\lambda_g}$$

$$\sim (\lambda_h - \varepsilon)\left(\frac{1}{2p}\right)^{\lambda_g} \log M(p^r, g) \qquad \ldots (7)$$

for a sequence $r = r_n \to \infty$ as $n \to \infty$.

In particular taking $p = 1/2$ in (7) and using (6) it follows that for a sequence $r = r_n \to \infty$,

$$\frac{\log \log M((r/2), f \circ g)}{\log \log M(r, h \circ g)} < \frac{\rho_h \log M((r/2), s)}{(\lambda_h - \varepsilon) \log M((r/2), g)}.$$

Thus

$$\liminf_{r \to \infty} \frac{\log \log M((r/2), f \circ g)}{\log \log M(r, h \circ g)} \leq \frac{\rho_h}{\lambda_h}.$$

Proof of Theorem 3—On using Theorem C, for sufficiently large r,

$$\log T(r, f \circ g) < (\rho_f + \varepsilon) \log M(r, g).$$

Now let $\rho_g(r)$ be a proximate order relative to $\log M(r, g)$. Then $\log M(r, g) \leq r^{\rho_g(r)}$ for all $r \geq r_0$. And let $0 < p < 1$. Then

$$\log T(r, f \circ g) \leq (\rho_f + \varepsilon) r^{\rho_g(r)} \text{ as } r \to \infty$$

$$= (\rho_f + \varepsilon) \frac{1}{p^{\rho_g}} (pr)^{\rho_g(pr)} \text{ as } r \to \infty$$

$$= (\rho_f + \varepsilon) \frac{1}{p^{\rho_g}} \log M(pr, g) \qquad \ldots (8)$$

for a sequence $r = r_n \to \infty$ as $n \to \infty$. Let $\alpha > 0$ be such that $\alpha^2 = p$. Then $\alpha < 1$ and so replacing R by r and r by αr in Theorem B we obtain for large r,

$$T(r, h \circ g) \geqslant \frac{1-\alpha}{1+\alpha} \log M(\alpha r, h \circ g)$$

$$\geqslant \frac{1-\alpha}{1+\alpha} \log M(c(\alpha) M(\alpha^2 r, g), h)$$

on using Theorem A, since $0 < \alpha < 1$.

Thus for a sequence $r = r_n (\geqslant r_0)$ it follows that

$$\log T(r_n, h \circ g) > \log \frac{1-\alpha}{1+\alpha} + (\lambda_h - \varepsilon) \log c(\alpha)$$

$$+ (\lambda_h - \varepsilon) \log M(\alpha^2 r_n, g). \qquad \ldots (9)$$

The result now follows from (8) and (9) and on making $p \to 1$.

Proof of Theorem 6—Using Theorem C it follows that for $r \geqslant r_0$,

$$\log \log T(r, f \circ g) < \log(\rho_f + \varepsilon) + (\rho_g + \varepsilon) \log r.$$

For the other inequality, we make use of the following inequality of Singh

$$T(r, h \circ k) > \frac{1}{3} \left\{ \frac{1}{9} M\left(\frac{r}{4}, k\right) \right\}^{\lambda_h - \varepsilon} \qquad \ldots (10)$$

for all $r \geqslant r_0$. Thus it follows that for the sequence $r\, r_n (\geqslant r_0)$,

$$\log T(r, h \circ k) > \frac{1}{2} (\lambda_h - \varepsilon) \left(\frac{r}{4}\right)^{\rho_k - \varepsilon}$$

so that

$$\log \log T(r, h \circ k) > \log \left\{ \frac{1}{2} (\lambda_h - \varepsilon) \right\} + (\rho_k - \varepsilon) \log \frac{r}{4}$$

$$\sim (\rho_k - \varepsilon) \log r \text{ as } r \to \infty.$$

It now follows that

$$\liminf_{r \to \infty} \frac{\log \log T(r, f \circ g)}{\log \log T(r, h \circ k)} \leqslant \frac{\rho_g}{\rho_k}.$$

Next, replacing $h \circ k$ by $f \circ g$ in (10) and considering the lower order of the right factor it is not difficult to see that

$$\log \log T(r, f \circ g) > \log \left\{\frac{1}{2}(\lambda_f - \varepsilon)\right\} + (\lambda_g - \varepsilon)\log \frac{r}{4}$$

for all $r \geqslant r_0$. Also for all $r \geqslant r_0$, we have

$$\log \log T(r, h \circ k) < \log(\rho_h + \varepsilon) + (\rho_k + \varepsilon)\log r.$$

And so

$$\liminf_{r \to \infty} \frac{\log \log T(r, f \circ g)}{\log \log T(r, h \circ k)} \geqslant \frac{\lambda_g}{\rho_k}.$$

Thus we have

$$\frac{\lambda_g}{\rho_k} \leqslant \liminf_{r \to \infty} \frac{\log \log T(r, f \circ g)}{\log \log T(r, h \circ k)} \leqslant \frac{\rho_g}{\rho_k}.$$

One can easily verify on the same lines that

$$\frac{\rho_g}{\rho_k} \leqslant \limsup_{r \to \infty} \frac{\log \log T(r, f \circ g)}{\log \log T(r, h \circ k)} \leqslant \frac{\rho_g}{\lambda_k}.$$

The theorem now follows since g and k are of regular growth.

Proof of Theorem 4—Using (10) it is easy to see that for a sequence $r = r_n \to \infty$

$$\frac{T(r, h \circ k)}{T(r, f \circ g)} > \frac{A \exp\{(\lambda_h - \varepsilon)(r/4)^{\rho_k - \varepsilon}\}}{\exp\{(\rho_f + \varepsilon) r^{\rho_g + \varepsilon}\}} \qquad \ldots (11)$$

where $A = 1/3\,(1/9)^{\lambda_h - \varepsilon}$. The right hand inequality tends to ∞ as $r \to \infty$, since we can choose $\varepsilon > 0$ such that $\rho_k - \varepsilon > \rho_g + \varepsilon$. We leave the details to the reader.

Proo of Theorem 5—Working as in Theorem 4,

$$\frac{T(r, h \circ k)}{T(r, \circ g)} > \frac{A \exp\{(\lambda_h - \varepsilon)(r/4)^{\lambda_k - \varepsilon}\}}{\exp\{(\rho_f + \varepsilon) r^{\rho_g + \varepsilon}\}}$$

for all $r \geqslant r_0$. The theorem follows since we can select $\varepsilon > 0$ such that $\lambda_k - \varepsilon > \rho_g + \varepsilon$.

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