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POINT-SET DOMINATION NUMBER OF A GRAPH

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A set D of vertices in a connected graph G is a 'psd-set' if for every set $S \subseteq V - D$ there exists a vertex $v \in D$ such that the subgraph $\langle S \cup \{v\} \rangle$ induced by $S \cup \{v\}$ is connected. The 'point-set domination number' $\gamma_p(G)$ of G is the minimum cardinality of a psd-set. Besides some bounds, exact values of $\gamma_p(G)$ are determined when G is a tree, block graph and cactus. A generalization of $\gamma_p(G)$ is also considered.

Let G be a connected graph. A set $D \subseteq V(G)$ is a point-set dominating set (psd-set) of G if for every set $S \subseteq V - D$, there exists a vertex $v \in D$ such that the subgraph $\langle S \cup \{v\} \rangle$ induced by $S \cup \{v\}$ is connected. The point-set domination number $\gamma_p(G)$ of G is the minimum cardinality of a psd-set.

A set $D \subseteq V(G)$ is a 'dominating set of' G if every vertex in $V - D$ is adjacent to some vertex in D. The 'domination number' $\gamma(G)$ of G is the minimum cardinality of a dominating set. As we can see from the bibliography[1], $\gamma(G)$ is a well-studied parameter.

The purpose of this paper is to obtain some bounds for $\gamma_p(G)$, and find its exact values for many classes of graphs including trees, block graphs and cactus. Further, we relate it to some other known domination parameters.

Henceforth, by a graph G we mean a connected graph, and V denotes its vertex set and $n = |V|$.

Clearly, if G has a vertex of degree $n-1$, then $\gamma_p(G) = 1$. Thus, $\gamma_p(K_n) = \gamma_p(K_{1,n}) = 1$. Also, $\gamma_p(K_{m,n}) = 2$ where $m \geq 2$, $n \geq 2$. For a path P_n and a cycle C_n on $n \geq 3$ vertices, $\gamma_p(P_n) = n - 2$, $\gamma_p(C_n) = 2$ for $n = 4$, 5, and $\gamma_p(C_n) = n - 2$ for $n \geq 6$.

We first obtain an elementary bound for γ_p.

Proposition 1 —For any graph G

$$\gamma_p \leq n - \Delta \qquad \ldots (1)$$

where Δ is the maximum degree of G.

PROOF: Let S be the set of vertices adjacent to a vertex of maximum degree in G. Then $V - S$ is a psd-set, and (1) follows.

The following result, though trivial, is quite useful in our further work.

Proposition 2—Let D be a psd-set of a graph G, and $u, v \in V-D$. Then $d(u, v) \leq 2$.

PROOF: Let $S=\{u, v\}$. There exists a vertex x in D such that the subgraph $\langle\{u, v, x\}\rangle$ is connected. This implies $d(u, v) \leq 2$.

We now obtain the major result of this paper from which we can deduce the exact values of γ_p for trees, block graphs and cactus.

By a γ_p-set we mean a minimum psd-set.

Proposition 3—Let G be a graph with cut vertices. Then

$$\gamma_p = \min \{n-\Delta, n-k\} \quad \ldots (2)$$

where $k=\max \{|V(B)|-\gamma_p(B)\}$, the maximum is taken over all the blocks B of G.

PROOF: Let D be a γ_p-set of G. We consider two cases.

Case 1: The set $V-D$ contains vertices not belonging to the same block—Let u and v be vertices in $V-D$ belonging to different blocks. Then both of them are adjacent to a cut vertex $w \in D$, for otherwise $d(u, v)>2$, which is not true by Proposition 2. Further, in this case $V-D$ contains only vertices belonging to the blocks at w. For, suppose there exists a vertex x in $V-D$ which belongs to a block at a cut vertex $w' \neq w$. Then $d(x, u)>2$ or $d(x, v)>2$ which is not true by Proposition 2. This proves that all vertices in $V-D$ are adjacent to the cut vertex w and $\gamma_p \geq n-d(w) \geq n-\Delta$, where $d(w)$ is the degree of w. Now, by (1), $\gamma_p = n-\Delta$.

Case 2: All vertices in $V-D$ belong to a single block B of G—In this case, clearly $\gamma_p(G)=n-(|V(B)|-\gamma_p(B)) \geq n-k$.

We note that a set S of vertices in G consisting of all vertices in all but one block F of G together with a minimum psd-set of F constitutes a psd set of G. Hence, $\gamma_p(G) \leq n-k$. Thus $\gamma_p(G)=n-k$, and (2) follows.

Corollary 3.1—For a tree, T, $\gamma_p = n-\Delta$.

PROOF : Since every block of T is K_2 we have $k=1$ and $\gamma_p = \min \{n-\Delta, n-1\} = n-\Delta$.

A block graph is one in which every block is complete and in a 'cactus' every block is either K_2 or a cycle.

Corollary 3.2—If G is a block graph or a cactus ($\neq C_5$), then $\gamma_p = n-\Delta$.

PROOF: Let G be a block graph. Then $k=\max \{|V(B)|-\gamma_p(B)\} = m-1$, where m is the order of a maximum clique. Since $\Delta \geq m-1$, we have $\gamma_p = \min \{n-\Delta, n-m+1\} = n-\Delta$.

Suppose now G is a cactus ($\neq C_5$). If $\Delta=2$, then G is either a path or a cycle. In either case, $\gamma_p = n-2 = n-\Delta$. Let $\Delta \geq 3$ and C_r be any cycle in G. Then, since G may contain a 5-cycle, $|V(C_r)|-\gamma_p(C_r) \leq r-(r-3)=3$, so $k \leq 3 \leq \Delta$.

Hence $\gamma_p = \min\{n-\Delta, n-k\} = n-\Delta$.

We now relate γ_p with some other known domination parameters.

In a graph G, a set $D \subseteq V$ is a set-dominating set (sd-set) if for every set $T \subseteq V-D$, there exists a nonempty set $S \subseteq D$ such that the subgraph $\langle S \cup T \rangle$ is connected. The set-domination number γ_s of G is the minimum cardinality of as sd-set (see Sampathkumar and Pushpa Latha[4]). The connected domination number γ_c of G is the minimum cardinality of a dominating set D such that the subgraph $\langle D \rangle$ is connected[2,3].

Clearly, any psd-set is a sd-set, and any sd-set is dominating set. Hence we have

$$\gamma \leq \gamma_s \leq \gamma_p. \qquad \ldots (3)$$

It is known that $\gamma_c - 1 \leq \gamma_s$ and diam $G - 1 \leq \gamma_s$ (see Sampathkumar and Pushpa Latha[4]). Thus from (3), $\gamma_c - 1 \leq \gamma_p$ and diam $G - 1 \leq \gamma_p$.

If a graph G has an independent sd-set then diam $G \leq 4$ (see Sampathkumar and Pushpa Latha[4]). A similar result holds for a psd-set.

Proposition 4—If a graph G has an independent psd-set D, then diam $G \leq 4$.

PROOF: If $u, v \in V-D$ then $d(u, v) \leq 2$ by Proposition 2. Suppose now $u \in D$ and $v \in V-D$. Since G is connected and D is independent, u is adjacent to some $w \in V-D$. Now, $d(u, v) \leq d(u, w) + d(w, v) \leq 3$ since $d(w, v) \leq 2$. Suppose now $u, v \in D$. As above, there exist $w_1, w_2 \in V-D$ such that u is adjacent to w_1 and v is adjacent to w_2. Now $d(u, v) \leq d(w_1, w_2) + 2 \leq 4$, since $d(w_1, w_2) \leq 2$. This proves diam $G \leq 4$.

The above bound is strict. For example, let G be a graph with exactly two blocks each block being C_4. Then diam $G = 4$ and the set of vertices $\{u, v, w\}$ of G where $d(u, v) = 4$ and w is the cut vertex, form an independent psd-set.

We now obtain a necessary condition for $\gamma = \gamma_p$.

Proposition 5—If $\gamma(G) = \gamma_p(G)$, then diam $G \leq 4$.

PROOF: Let $\gamma(G) = \gamma_p(G)$. Then every γ_p-set of G is a γ-set. Let D be a γ_p-set. Since D is also a γ-set, every $v \in D$ is adjacent to at least one vertex in $V-D$. As in the proof of Proposition 4, one can show that $d(u, v) \leq 4$ for all $u, v \in V$.

r-Point Set-Domination

We now consider a generalization of point-set domination.

Let $r \geq 1$ be an integer. A set $D \subseteq V$ is a r-point set-dominating set (r-psd-set) of a graph G if for every set $T \subseteq V-D$ there exists a nonempty set $S \subseteq D$ containing at most r vertices such that the subgraph $\langle S \cup T \rangle$ is connected. The r-point set-domination number $\gamma_{rp}(G)$ of G is the minimum cardinality of a r-psd-set.

We observe that every r-psd-set is a dominating set and $\gamma_{1p} = \gamma_p$. Also, for $1 \leq t \leq r$, $\gamma_{rp} \leq \gamma_{tp} \leq \gamma_{1p} = \gamma_p$.

Since every r-psd-set is a set dominating set, we have $\gamma_1 \leq \gamma_{rp}$. If $r \geq \gamma_1$, then γ_1-set is also a r-psd-set. Hence $\gamma_{rp} \leq \gamma_1$. Thus, for $r \geq \gamma_1$, we have $\gamma_p = \gamma_{rp}$.

We observe that if G is a path of order n, then $\gamma_{rp} = n-2$. For a cycle C_n we have $\gamma_{rp}(C_3) = 1$, $\gamma_{rp}(C_4) = 2$, and $\gamma_{rp}(C_n) = n-3$ for $5 \leq n \leq 2r+3$, and $\gamma_{rp}(C_n) = n-2$ for $n > 2r+3$.

For a vertex v, let $N(v)$ be the set of all neighbours of v, and for a set $S \subseteq V$,

let $$N(S) = \bigcup_{v \in S} N(v). \qquad \ldots (4)$$

Let $k = \max \{|N(S) \cap (V-S)|\}$ where the maximum is taken over all sets S such that

a) $|S| \leq r$, and

b) the subgraph $\langle S \rangle$ is connected.

The following result generalizes Proposition 1.

Proposition 6—For any graph G of order n,

$$\gamma_{rp} \leq n-k \qquad \ldots (5)$$

Proof: Let S be a set of vertices satisfying the conditions (a) and (b). Then, clearly $V-(N(S)-S)$ is a r-psd-set, and (5) follows

We now show that the bound in (5) is attained for trees.

Proposition 7—For a tree T with n vertices,

$$\gamma_{rp} = n-k \qquad \ldots (6)$$

Proof: Let D be a γ_{rp}-set (i.e. a minimum r-psd-set). If T is a path then $\gamma_{rp} = n-2 = n-k$. Suppose now $\Delta(T) \geq 3$. Then since $\gamma_{rp} \leq \gamma_p \leq n-\Delta$, there exists at least three vertices, say x, y and z in $V-D$. The set $V-D$ is independent. For, suppose x is adjacent to y. Then either the $x-z$ path contains y, in which case we let $T = \{x,z\}$, or the $y-z$ path contains x, in which case we let $T = \{y, z\}$. In both cases, however, there is no subset S of D such that the subgraph $\langle S \cup T \rangle$ is connected. This contradicts the fact that D is a r-psd-set. Hence $V-D$ is independent. Now, let S be a subset of D with minimum number of vertices such that the subgraph $T' = \langle S \cup$.

We claim that $\deg v_i = 1$ for each vertex $v_i \in V-D$ in T'. For, suppose u_1 and u_2 are two vertices in S adjacent to $v_1 \in V-D$. Let $V-D = \{v_1, v_2, \ldots, v_s\}$. Then clearly, u_1 is on some $v_1 - v_i$ path in T'. For, otherwise the union of all $v_1 - v_i$ paths, $2 \leq i \leq s$, in T' forms a connected graph not containing u_1, contradicting the minimality of S. Similarly, u_2 is on some $v_1 - v_j$ path in T' where $i \neq j$.

There is no set $S_1 \subseteq D$ such that the subgraph $\langle S_1 \cup \{v_i, v_j\}\rangle$ is connected, since the only $v_i - v_j$ path in T contains v_1, and $v_1 \notin D$. This contradicts the fact that D is a γ_{rp}-set. Hence, each vertex of $V - D$ has degree one in T, and $v_1, v_2, \ldots v_k$ are end vertices of T. Removing these end vertices in T, we see that the subgraph $\langle S\rangle$ is connected, and each vertex of $V - D$ is adjacent to some vertex in S. Now, since $S \subseteq D$ and $V - D \subseteq N(S)$ it follows that

$$|V - D| = |N(S) \cap V - D| \leq |N(S) \cap V - S| \leq k.$$

Hence $\gamma_{rp} \geq n - k$. Now, (6) follows from (5).

Note that Proposition 7 generalizes Corollary 3.1.

Remark: In a tree observe that $|N(S) \cap V - S| = \sum_{v_i \in S} d(v_i) - 2r + 2$, where $|S| = r$. For, the subtree $\langle S\rangle$ has $r - 1$ edges, and the sum of the degrees in $\langle S\rangle$ is $2(r - 1)$.

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References

1. S.T. Hedetniemi and R.C. Laskar, *Discrete Math.* **86** (1990), 257-77.
2. S.T. Hedetniemi and R. Laskar, *Graph Theory and Combinatorics*, Ed. B. Bollobas, Academic Press, London, 1984, pp 209-18.
3. E. Sampathkumar and H.B. Walikar, *J. Math. Phy. Sci.*, **13** (1979), 607-13.
4. E. Sampathkumar and L. Pushpa Latha, Set domination in graphs, *J. Graph Theory* to appear.