

A METHOD FOR THE APPROXIMATE SOLUTIONS OF THE UNSTEADY BOUNDARY LAYER EQUATIONS

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The approximate integral method proposed by Bianchini *et al*¹ to solve the unsteady boundary layer equations is considered here with a simple modification to the scale function for the similarity variable. This is done by introducing a time dependent length scale. The closed form solutions, thus obtained, give satisfactory results for the velocity profile and the skin friction to a limiting case in comparison with the results of the past investigators.

1. INTRODUCTION

Bianchini *et al*¹ introduced an approximate method to calculate the characteristics of the boundary layer flows, taking the potential flow simply as a function of time. Later, this first approximate method has been developed by Socio and Pozzi⁷ with a rigorous mathematical approach. The potential flow in this case is taken to be a function of x and t . The essence of the first approximate method is to assume a similarity solution even in those situations where similarity solutions do not exist, and to find a suitable scale factor for the similarity variable. In comparison with the classical series and numerical solutions and other approximate methods³ which also heavily rely on numerical computations, the above methods do not require lengthy calculations. Moreover, no linearization is required. A successful application of the first approximate method has recently been done by Palekar and Sarma⁴ to the case of a steady boundary layer flow with suction and blowing.

In the present paper, the method of Bianchini *et al*¹ is modified by introducing a time dependent length scale along with the scale function for the similarity variable. The first order differential equation that Bianchini *et al* obtained for the scale function is obtained here in a much more simpler form which leads to a simple solution. The velocity profiles and the skin friction to a limiting case are thus obtained and compared with the classical results.

2. THE GOVERNING EQUATIONS AND THE METHOD OF SOLUTION

The basic two-dimensional equations governing the unsteady laminar boundary-layer over a semi-infinite flat plate are⁶;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \dots (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{dU}{dt} + \nu \frac{\partial^2 u}{\partial y^2} \quad \dots (2)$$

where x and y are the cartesian coordinates along the plate and normal to it, respectively, t is the time, u and v are the velocity components along x and y , and $U(t)$ is the potential flow far away from the plate.

The boundary conditions associated with the above system will be considered as

$$u = v = 0 \quad \text{at} \quad y = 0 \quad \dots (3)$$

$$u = U(t) \quad \text{as} \quad y \rightarrow \infty.$$

Now, following Bianchini *et al.*, the velocity profile is assumed to have the form

$$u = U(t)f(\eta) \quad \dots (4)$$

$$\text{with } \eta = Y/h(X) \quad \dots (5)$$

$$\text{where } Y = y/\sigma \quad \text{and} \quad X = x/\sigma \quad \dots (6)$$

and $\sigma = \sigma(t)$ is a length scale but a function of time.

Here eqns. (5) and (6) constitute the point of departure from the assumption of Bianchini *et al.*

Integrating eqn. (2) across the boundary layer from zero to infinity, using the continuity equation (1) and the assumptions (5) and (6), it is obtained

$$\begin{aligned} \frac{\sigma^2}{\nu} \frac{1}{U} \frac{dU}{dt} a_1 h + \frac{\sigma}{\nu} \frac{d\sigma}{dt} \left[X \frac{dh}{dX} - h a_2 \right] + \\ + \frac{U\sigma}{\nu} a_3 \frac{dh}{dX} = \frac{a_4}{h} \quad \dots (7) \end{aligned}$$

where

$$\left. \begin{aligned} a_1 &= \int_0^\infty (1-f)d\eta, \quad a_2 = \int_0^\infty \eta f d\eta \\ a_3 &= \int_0^\infty f(1-f)d\eta, \quad a_4 = \left(\frac{\partial f}{\partial \eta} \right)_{\eta=0} \end{aligned} \right\} \quad \dots (8)$$

In order to obtain a simple solution of (7), one can try a class of solutions of (7) by setting

$$\frac{\sigma^2}{\nu} \frac{1}{U} \frac{dU}{dt} = -\frac{\sigma}{\nu} \frac{d\sigma}{dt} \quad \dots (9)$$

Thus integrating (9), one obtains

$$U\sigma = \text{constant} = A; \text{ or } U(t) = A\sigma^{-1}. \quad \dots (10)$$

Now taking $A = U_0\sigma_0$, where U_0 and σ_0 are the reference values of U and σ at $t = t_0$, one obtains the reference Reynolds number R as

$$R = U\sigma/\nu = U_0\sigma_0/\nu = R_0.$$

Therefore, eqn. (7) can be written as

$$\begin{aligned} -\frac{\sigma}{\nu} \frac{d\sigma}{dt} \alpha_1 h + \frac{\sigma}{\nu} \frac{d\sigma}{dt} \left(X \frac{dh}{dX} - h \right) \alpha_2 + \\ + R_0 \alpha_3 \frac{dh}{dX} = \frac{\alpha_4}{h}. \end{aligned} \quad \dots (11)$$

The two-dimensional boundary layer equations (1) and (2) are now reduced to a single first order differential equation for the scale function $h(X)$ except for the term $\sigma/\nu d\sigma/dt$ where time t appears explicitly. But, since the notion of the present problem is to obtain a similar solution, the similarity condition requires that $\sigma/\nu d\sigma/dt$ must be a constant. Hence it is supposed that

$$\frac{\sigma}{\nu} \frac{d\sigma}{dt} = K \quad \dots (12)$$

where K is an arbitrary constant.

Now integrating (12), one obtains

$$\sigma = \sqrt{2K\nu t} \quad \dots (13)$$

where the constant of integration is determined in such a way that $\sigma = 0$ when $t = 0$.

Hence eqn. (11) can finally be written as

$$-K\alpha_1 h + K\alpha_2 \left(X \frac{dh}{dX} - h \right) +$$

$$R_0 a_3 \frac{dh}{dX} = \frac{a_4}{h} \dots (14)$$

An integrating to the above equation gives the scale parameter and hence a solution for the velocity profiles. The constant of integration arising out of this integration will be determined by adding an initial condition to the set of boundary conditions adopted in (4).

3. THE GENERAL SOLUTION

In order to have the general solution to the scale parameter it is now necessary to make an assumption of the function $f(\eta)$ for the velocity distribution proposed in (5).

A convenient choice for $f(\eta)$ is

$$f(\eta) = \text{erf}(\eta) \dots (15)$$

Hence, from (8) and (13), one has

$$a_1 = \frac{1}{\sqrt{\pi}}, \quad a_2 = \frac{1}{\sqrt{\pi}}, \quad a_3 = (\sqrt{2}-1)/\sqrt{\pi}, \quad a_4 = \frac{2}{\sqrt{\pi}} \dots (16)$$

With the above values, a general solution of eqn. (14) subject to the initial condition

$$h(X) = 0 \quad \text{when} \quad X = 0$$

is obtained as

$$\log\left(1 + \frac{\gamma h^2}{a_4}\right) = \beta \log(1 + \delta X) \dots (17)$$

where

$$\delta = \frac{K a_2}{R_0 a_3}, \quad \gamma = K(a_1 + a_2) \quad \text{and} \quad \beta = \frac{2\theta\gamma}{K a_2}$$

Now from (17) it is apparent that h is constant for a fixed X . Thus, taking $h = \text{constant} = C$ for a fixed X , the velocity profile is obtained from (4), (5) and (15) as

$$u' = u/U(t) = \text{erf}\{y/\sigma c\} \dots (18)$$

But since $\sigma = \sqrt{2K\nu t}$, taking $K = 2$, (18) can be written as

$$u' = u/U(t) = \text{erf}(y/2c\sqrt{\nu t}) \dots (19)$$

where, from (10), $U(t) = A\sigma_{-1} = A(2\nu t)^{-1/2}$.

Thus $U(t) \propto t^{-1/2}$ (20)

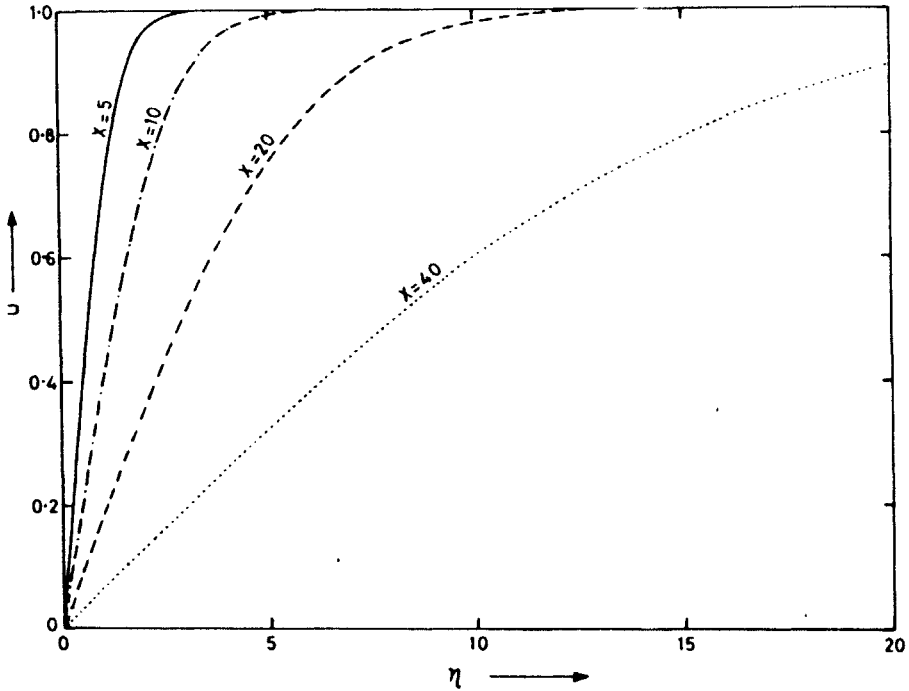
The solution (19) can be validated from an observation of the exact solution

$$u' = u/U(t) = \text{erf}(y/2\sqrt{\nu t}) \quad \dots (21)$$

where $U(t) \propto t^{-\alpha}$ (α being a constant).

The above solution is the well known exact solution to the two-dimensional unsteady boundary layer problem (Rosenhead, p. 360).

However, with the solution of $h(X)$ from (17), velocity profiles for various values of X are obtained and shown in Fig. 1. It is apparent from Fig. 1 that the velocity profiles become fuller as one approaches the leading edge, or otherwise as expected, the boundary layer thickness increases with the increase of the downstream position.



4. APPROXIMATE SOLUTIONS

From eqn (17) it follows that $h(X) \rightarrow 0$ as $X \rightarrow 0$. Thus for small X , one can consider $\gamma h^2/a_4 < 1$. Hence, expanding both sides of (17), and subsequently neglecting the terms of higher powers of $\gamma h^2/a_4$ and δX , one obtains

$$\gamma h^2/a_4 = \beta \delta X$$

or

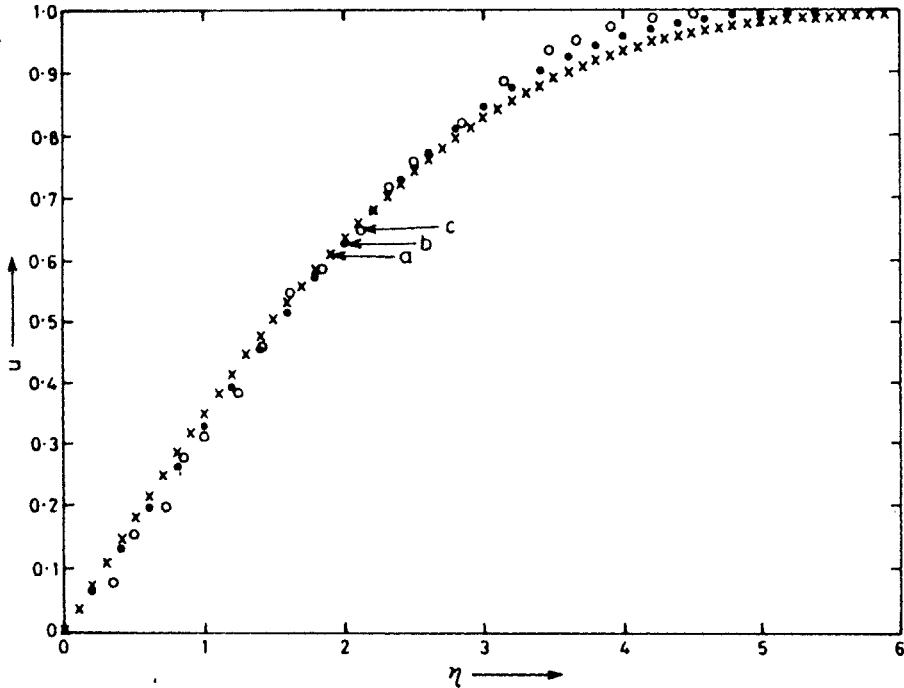
$$h(X) = (\alpha_4 \beta \delta X / \gamma)^{1/2}$$

Therefore, from (15), the velocity profiles for small X are obtained as

$$u' = \operatorname{erf} \left(z / \sqrt{\frac{\alpha_4 \beta \delta}{\gamma}} \right) \quad \dots (22)$$

where $z = y / \sqrt{X}$.

A comparison for this result is made in Fig. 2 with those of Blasius's exact solution and the experiments of Hill and Stenning₂. It shows quite a good agreement.



Again, in order to provide a good approximate solution of the steady boundary layer flow over a flat plate, it is assumed that $\sigma = L$ and hence $K = 0$ in equation (14). Then one obtains the solution of $h(X)$ as

$$h = \sqrt{\frac{2 \alpha_4}{R_0 \alpha_3}} X$$

Thus

$$\left(\frac{\partial u}{\partial y} \right)_{y=0} = \frac{U_0}{L h(X)} \frac{2}{\sqrt{\pi}}$$

Therefore, the skin friction denoted by C_f , is obtained as

$$\begin{aligned} \frac{1}{2} C_f &= \frac{\tau_w}{\rho U_0^2} = \sqrt{\frac{2 a_3}{\pi a_4}} \left(\frac{\nu}{U_0 x} \right)^{1/2} \\ &= .363109 \left(\frac{\nu}{U_0 x} \right)^{1/2} \end{aligned}$$

5. CONCLUSION

Following Bianchini *et al.* a method has been proposed for the solution of the unsteady, incompressible boundary layer problem which reduces the original two dimensional partial differential equations to simple ordinary first order differential equation. The method does not require rigorous mathematical approach, rather a much more simple form of solution has been obtained in comparison with those of Bianchini *et al.*¹ and Socio and Pozzi⁷. With this simple form of solutions, one can attain the well known exact solution of the problem under consideration for a fixed downstream position by taking the weight factor of the length scale, $K=2$. Due to this reason all the other calculations have been done with $K=2$. One can, certainly, calculate the velocity profiles with other values of K to ascertain the variation of the boundary layer thickness with K . From the calculation it is also observed that the usual Blasius' similar velocity profile is valid for small X , but breaks down otherwise. Nevertheless, the calculation of the skin friction in the limit situation of steady flow corresponds well with Blasius' exact formula. Therefore, as far as the simplicity is concerned, the present method is advantageous over the classical and other time consuming numerical methods. To test further capability of the method works are under progress.

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