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ASYMPTOTIC AND OSCILLATORY BEHAVIOR OF SOLUTIONS OF NONLINEAR SECOND ORDER DIFFERENCE EQUATIONS

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The asymptotic and oscillatory behavior of solutions of nonlinear second order difference equations of the form

$$\Delta (a_n \Delta y_n) + p_n \Delta y_n + q_n f(y_n) = 0, \quad n \in N$$

is studied. Examples are inserted to illustrate the results

1. INTRODUCTION

In this paper we are concerned with the nonlinear difference equation of the type

$$\Delta (a_n \Delta y_n) + p_n \Delta y_n + q_n f(y_n) = 0, \quad n \in N \qquad \ldots (1)$$

where Δ is the operator defined by the equation $\Delta y_n = y_{n+1} - y_n$ and the set $N = \{ 0, 1, 2, \ldots\}$. Throughout we shall assume the following conditions without further mention :

(c_1) $a_n > 0$, $p_n \geq 0$ and $q_n > 0$ for all $n \geq n_0 \in N$

(c_2) $f : R \to R = (-\infty, \infty)$; $u f(u) > 0$ for $u \neq 0$.

By a solution of (1) we mean a real sequence $\{y_n\}$, $n \in N$ satisfying (1). We consider only such solutions which are nontrivial for all large n. A solution of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory.

In recent years there has been an increasing interest in the study of the qualitative behavior of solutions of difference equations of type (1) and/or related equations[1-9].

In the study of difference equation

$$\Delta (a_n \Delta y_n) + q_n f(y_n) = 0$$

many criteria for oscillation exist with a common restriction, namely the condition $\sum \frac{1}{a_n} = \infty$ is required on the sequence $\{a_n\}$.

As examples to this study we cite the papers of Kulenovic and Budincevic[3] and Szmanda[6, 8].

Our purpose in this paper is to study the asymptotic and oscillatory behavior of solutions of eqn. (1), where conditions on the sequences $\{a_n\}$ and $\{p_n\}$ are different from those imposed in Kulenovic and Budincevic[3] and Szmanda[6, 8].

2. Main Results

We begin with the following theorem.

Theorem 1 — Assume that

(i) the function f is continuous and nondecreasing on R

(ii) $a_n - p_n \geq 0$ for $n \geq n_0 \in N$ ... (2)

(iii) there exists a positive sequence $\{h_n\}$ such that

$$\Delta h_n \leq 0, \quad \Delta (p_n h_n) \leq 0 \quad \text{and} \quad \Delta (a_{n+1} \Delta h_n) \geq 0 \qquad \ldots (3)$$

for all $n \geq n_0 \in N$.

If $$\sum^{\infty} h_n q_n = \infty \qquad \ldots (4)$$

and

$$\sum^{\infty} \frac{1}{a_n h_n} \sum_{s=n_0}^{n-1} h_s q_s = \infty \qquad \ldots (5)$$

then every solution of eqn. (1) is either oscillatory or tends to zero monotonically as $n \to \infty$.

Proof : Let $\{y_n\}$ be a nonoscillatory solution of eqn. (1) which must then eventually of constant sign. Without loss of generality, we assume that $y_n > 0$ for all $n \geq n_0 \in N$. Now we consider the following cases for the behavior of $\{ \Delta y_n \}$.

Case 1 — (a) $\{ \Delta y_n \}$ is oscillatory. First, suppose there exists some integer $n_1 \geq n_0$ such that $\Delta y_{n_1} < 0$. Now from (1)

$$\Delta (a_{n_1} \Delta y_{n_1}) \Delta y_{n_1} = - p_{n_1} (\Delta y_{n_1})^2 - q_{n_1} f(y_{n_1}) \Delta y_{n_1} > - p_{n_1} (\Delta y_{n_1})^2$$

since $-q_{n_1} f(y_{n_1}) \Delta y_n > 0$. Hence

$$\Delta y_{n_1} [a_{n_1+1} \Delta y_{n_1+1} - a_{n_1} \Delta y_{n_1}] > -p_{n_1} (\Delta y_{n_1})^2$$

or $$a_{n_1+1} \Delta y_{n_1+1} \Delta y_{n_1} > (a_{n_1} - p_{n_1}) (\Delta y_{n_1})^2 \geq 0.$$

Thus, by dividing the above by negative term $a_{n_1+1} \Delta y_{n_1}$ we obtain

$$\Delta y_{n_1+1} < 0.$$

By induction, we obtain $\Delta y_n < 0$ for all $n \geq n_1$, a contradiction. Thus it must be the case that $\Delta y_n \geq 0$ for all $n \geq n_0$.

(b) Suppose for some n_1, $\Delta y_{n_1} = 0$. Then eqn. (1) implies $\Delta y_{n_1+1} < 0$, and we obtain case 1 (a). Thus $\Delta y_n > 0$ for all $n \geq n_0$. However, this contradicts the assumption that $\{ \Delta y_n \}$ oscillates.

Case 2 — $\Delta y_n > 0$ for all $n \geq n_1$ for some integer $n_1 \geq n_0 \in N$. We define

$$z_n = \frac{h_n v_n}{f(y_n)}$$

where $v_n = a_n \Delta y_n$. Note that $z_n > 0$.

Then for $n \geq n_1$

$$\Delta z_n = -h_n q_n - \frac{p_n h_n \Delta y_n}{f(y_n)} + \frac{\Delta h_n v_{n+1}}{f(y_{n+1})} - \frac{h_n v_{n+1} \Delta f(y_n)}{f(y_n) f(y_{n+1})} \quad \ldots (6)$$

From the hypothesis, we obtain

$$\Delta z_n \leq -h_n q_n \quad \text{for} \quad n \geq n_1.$$

Summing the above inequality from n_1 to n, we have

$$\sum_{s=n_1}^{n} h_s q_s \leq z_{n_1} - z_{n+1} \leq z_{n_1} < \infty$$

which contradicts condition (4).

Case 3 — $\Delta y_n < 0$ for $n \geq n_1$ for some integer $n_1 \geq n_0$. Since we are assuming $y_n > 0$, it follows that $\lim_{n \to \infty} y_n = b$, $b \geq 0$. We clain that $b = 0$. To prove it, assume $b > 0$ and define

$$u_n = h_n a_n \Delta y_n \quad \text{for} \quad n \geq n_1.$$

We then obtain for $n \geq n_1$.

$$\Delta u_n = -h_n p_n \Delta y_n - h_n q_n f(y_n) + (\Delta h_n)(a_{n+1} \Delta y_{n+1}). \quad \ldots (7)$$

Hence, for all $n \geq n_1$, we have

$$u_n = u_{n_1} - \sum_{s=n_1}^{n-1} p_s h_s \Delta y_s - \sum_{s=n_1}^{n-1} h_s q_s f(y_s) + \sum_{s=n_1}^{n-1} (\Delta h_s)(a_{s+1} \Delta y_{s+1})$$

$$= u_{n_1} - \sum_{s=n_1}^{n-1} p_s h_s \Delta y_s - f(y_n) \sum_{s=n_1}^{n-1} h_s q_s + \sum_{s=n_1}^{n-1} (\Delta f(y_s)) \left(\sum_{t=n_1}^{s} h_t q_t \right) + \sum_{s=n_1}^{n-1} (\Delta h_s)(a_{s+1} \Delta y_{s+1}).$$

Since $\Delta f(y_n) \leq 0$ for $n \geq n_1$, using (3) we have for $n \geq n_1$

$$u_n \leq u_{n_1} + p_{n_1} h_{n_1} y_{n_1} - f(y_n) \sum_{s=n_1}^{n-1} h_s q_s - a_{n_1+1} (\Delta h_{n_1}) y_{n_1+1}.$$

So for every $n \geq n_1$

$$u_n \leq M - f(b) \sum_{s=n_1}^{n-1} h_s q_s$$

where $M = u_{n_1} + p_{n_1} h_{n_1} y_{n_1} - a_{n_1+1} (\Delta h_{n_1}) y_{n_1+1}$.

By assumption (4) there exists an integer $n_2 \geq n_1$ such that

$$u_n \leq -\frac{f(b)}{2} \sum_{s=n_1}^{n-1} h_s q_s \quad \text{for} \quad n \geq n_2.$$

Thus

$$\sum_{s=n_2}^{n-1} \Delta y_s \leq -\frac{f(b)}{2} \sum_{s=n_2}^{n-1} \frac{1}{a_s h_s} \left(\sum_{t=n_1}^{s-1} h_t q_t \right)$$

By condition (5), we get

$$y_n \to -\infty \quad \text{as} \quad n \to \infty$$

a contradiction to the fact that $y_n > 0$ for $n \geq n_0$.

Thus $b = 0$ and $y_n \to -\infty$ as $n \to \infty$.

In the following theorem we study the oscillatory behavior of eqn. (1) subject to the conditions

$$\int^{+\infty} \frac{du}{f(u)} < \infty \qquad \text{and} \qquad \int^{-\infty} \frac{du}{f(u)} < \infty \qquad \ldots (8)$$

Theorem 2 — Suppose that the conditions (i), (ii) and (8) hold. Assume that there exists a positive sequence $\{ h_n \}$ such that

$$\Delta h_n \geq 0, \ \Delta (p_n h_n) \leq 0 \text{ and } \Delta (a_n \Delta h_n) \leq 0, \text{ for } n \geq n_0 \in N \qquad \ldots (9)$$

If conditions (4) and (5) hold then every solution of eqn. (1) is either oscillatory or tends to zero monotonically as $n \to \infty$.

PROOF : Let $\{y_n\}$ be a nonoscillatory solution of eqn. (1). Assume that $y_n > 0$ for $n \geq n_0 \in N$, as in the proof of Theorem 1. Here we consider three cases of the behaviour of $\{\Delta y_n\}$. The case when $\{\Delta y_n\}$ is oscillatory is similar to Case 1 in Theorem 1. For Case 2, that is when $\Delta y_n > 0$ for $n > n_0 \in N$, we consider the function z_n defined in the proof of Theorem 1 (Case 2). Then for every $n \geq n_1$, we obtain (6) and using conditions (i) and (9) we get

$$\Delta z_n \leq - h_n q_n + \frac{(\Delta h_n) y_{n+1}}{f(y_{n+1})}$$

Since $y_{n+1} \leq y_n$ and $(a_n \Delta h_n)$ is nonincreasing for $n \geq n_1$, we have

$$\Delta z_n \leq - h_n q_n + a_{n_1} \Delta h_{n_1} + \frac{\Delta y_n}{f(y_{n+1})}, \ n \geq n_1 . \qquad \ldots (10)$$

Now, for $y_n \leq x \leq y_{n+1}$, we have $\frac{1}{f(x)} \geq \frac{1}{f(y_{n+1})}$, and it follows that

$$\int_{y_n}^{y_{n+1}} \frac{dx}{f(x)} \geq \frac{\Delta y_n}{f(y_{n+1})} .$$

Using the above inequality in (10) and summing the resulting inequality from n_1 to n leads to

$$\sum_{s=n_1}^{n} h_s q_s \leq z_{n_1} - z_{n+1} + a_{n_1} \Delta h_{n_1} \int_{y_{n_1}}^{y_{n+1}} \frac{dx}{f(x)} .$$

The above inequality in view of (8) and $z_n \geq 0$, $n \geq n_1$, gives

$$\sum_{s=n_1}^{n} h_s q_s < \infty$$

which contradicts (4). Next, for Case 3, we use the function u_n considered in the proof of Theorem 1 (Case 3) and obtain (7).

Then by condition (9) we get

$$\Delta u_n \leq - h_n p_n \Delta y_n - h_n q_n f(y_n) , \quad n \geq n_1 .$$

The rest of the proof is similar to that of Theorem 1 (Case 3) and hence is omitted.

Example 1 — Consider the difference equation

$$\Delta (n^2 \Delta y_n) + \frac{3}{n} \Delta y_n + \frac{n(n^2 + 3n + 6)}{(n+1)(n+2)} y_n^3 = 0, \quad n \geq 2. \qquad \ldots (11)$$

The conditions of Theorem 2 are satisfied for $h_n = 1$. eqn. (11) has a nonoscillatory solution $y_n = \frac{1}{n} \to 0$ as $n \to \infty$.

Example 2 — The difference equation

$$\Delta (n^3 \Delta y_n) + \frac{1}{(n+2)} \Delta y_n + \frac{n(n^2 + 3n + 1) + 1}{(n+1)(n+2)} y_n = 0, \; n \geq 1 \qquad \ldots (12)$$

has a nonoscillatory solution $y_n = \frac{1}{n} \to 0$ as $n \to \infty$. The conditions of Theorem 1 are satisfied if we choose $h_n = 1$.

Note that Theorem 2 is not applicable to eqn. (12), since condition (8) is violated.

Finally we discuss the oscillatory and asymptotic behavior of eqn. (1) subject to the condition

$$f(v) = \alpha v \qquad \ldots (13)$$

where α is a positive constant. Note that (13) implies that equation (1) is linear.

Theorem 3 — Suppose that conditions (i), (ii) and (13) hold. Assume that there exists a positive sequence $\{ h_n \}$ such that

$$\Delta h_n \geq 0 \quad \text{and} \quad \Delta (p_n h_n) \leq 0 \quad \text{for} \quad n \geq n_0 \in N. \qquad \ldots (14)$$

If condition (5) holds and

$$\sum^{\infty} \left(h_n q_n - \frac{a_n (\Delta h_n)^2}{4\alpha h_n} \right) = \infty \qquad \ldots (15)$$

then every solution of eqn. (1) is either oscillatory or tends to zero monotonically as $n \to \infty$.

PROOF : Let $\{ y_n \}$ be a nonoscillatory solution of eqn. (1). As in the proof of Theorem 1 assume that $y_n > 0$ for $n \geq n_0 \in N$. Further as in the proof of Theorem 1 (Case 1), $\{ \Delta y_n \}$ cannot oscillate, so we consider the other two cases. For Case 2, that is when $\Delta y_n > 0$ for $n \geq n_0 \in N$, we consider the function z_n defined in the proof of Theorem 1 (Case 2). We obtain (6) and using condition (13) and the fact that $p_n \geq 0$, we get

$$\Delta z_n \leq - h_n q_n + \frac{\Delta h_n v_{n+1}}{f(y_{n+1})} - \frac{\alpha h_n v_{n+1} \Delta y_n}{f(y_n) f(y_{n+1})}. \qquad \ldots (16)$$

Using the inequalities $v_{n+1} \leq v_n$ and $f(y_n) \leq f(y_{n+1})$, we obtain

$$\Delta z_n \leq - h_n q_n + \frac{\Delta h_n}{h_{n+1}} z_{n+1} - \frac{\alpha h_n}{a_n h_{n+1}^2} z_{n+1}^2$$

$$= - \left[h_n q_n - \frac{a_n (\Delta h_n)^2}{4\alpha h_n} \right] - \frac{\alpha h_n}{a_n h_{n+1}^2} \left[z_{n+1} - \frac{a_n \Delta h_n h_{n+1}}{2\alpha h_n} \right]^2$$

$$\leq - \left[h_n q_n - \frac{a_n (\Delta h_n)^2}{4\alpha h_n} \right], \quad n \geq n_1.$$

Summing the above inequality from n_1 to n, to obtain

$$\sum_{s=n_1}^{n} \left[h_s q_s - \frac{a_s (\Delta h_s)^2}{4\alpha h_s} \right] \leq z_{n_1} - z_{n+1} \leq z_{n_1} < \infty$$

which contradicts (15). The proof for the case $\Delta y_n < 0$ for $n \geq n_1 \geq n_0 \in N$ is similar to that of Case 3 of Theorem 2.

Example 3 — Consider the difference equation

$$\Delta (n^2 (n+1) \Delta y_n) + \Delta y_n + \frac{n^2 + n + 1}{n+1} y_n = 0, \quad n \geq 1. \qquad \ldots (17)$$

The hypotheses of Theorem 3 are satisfied for $h_1 = 1$ and $\alpha = 1$, and hence all the solutions of (17) are either oscillatory or tends to zero monotonically as $n \to \infty$. Equation (17) admits a solution $y_n = \frac{1}{n} \to 0$ as $n \to \infty$.

Remark : Since the condition $\sum^{\infty} \frac{1}{a_n} = \infty$ is not satisfied in equations given in Eamples 1-3, the results in Hooker and Patula[2], Szafranski[5] and Szmanda[7] are not applicable to eqns. (11), (12) and (17).

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