Indian J. pure appl. Math., 24(6) : 373-384, June 1993

ON CERTAIN NEW FINITE DIFFERENCE INEQUALITIES

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(Received 18 June 1992, accepted 11 March 1993)

In this paper we present some new finite difference inequalities which can be used as handy tools in the study of various problems in the theorey of certain classes of finite difference and sum-difference equations.

1. INTRODUCTION

Mate and Nevai[2] proved the following finite difference inequality.

Let $u(n) > 0$, $p(n) > 0$ be real-valued functions defined on integers and let $c > 0$ be a constant. If

$$u(n) \leq c + \sum_{s=n+1}^{x} P(s)\, u(s) \qquad \ldots (1)$$

then

$$u(n) \leq c \exp\left\{ \sum_{s=n+1}^{x} p(s) \right\} \qquad \ldots (2)$$

Finite difference inequalities of this type have many significant applications in the theory of finite difference equations, and they admit many variations and generalizations[1-10]. Although an enormous amount of attention has been given to the investigation of various types of finite difference inequalities, it appears that there are certain classes of finite difference and sum-difference equations for which the existing results on such inequalities do not seem to be directly applicable. In view of the important applications of the above inequality given by other workers[1, 2, 3, 7, 8], it seems reasonable to look for some new finite difference inequalities of the above type which would be equally important in certain applications. The main object of this paper is to establish some new finite difference inequalities which can be used in the analysis of certain classes of finite differnce and sum-difference equations for which earlier inequalities in the literature do not apply.

2. Main Results

In what follows we let $Z = \{..., -2, -1, 0, 1, 2, ...\}$ and use the notations m, n, p, q to denote the elements of Z. Let R denote the set of real numbers and $R_+ = [0, \infty)$ and $R_1 = [1, \infty)$ are subsets of R. For $t_1 > t_2$, $t_1, t_2 \in Z$ and any function $h : Z \to R$, we use the usual convention $\sum_{s=t_1}^{t_2} h(s) = 0$ and $\prod_{s=t_1}^{t_2} h(s) = 1$. Throughout, without further mention, we assume that all the sums and products converge on the respective domains of their definitions.

An interesting and useful finite difference inequality is established in the following theorem.

Theorem 1 - Let $u(n)$ and $f(n)$ be functions defined on Z into R_+ and $c \geq 0$ be a constant.

$$u^2(n) \leq c + \sum_{s=n+1}^{\infty} f(s)u(s) \qquad \ldots (3)$$

then

$$u(n) \leq \sqrt{c} + \frac{1}{2} \sum_{s=n+1}^{\infty} f(s). \qquad \ldots (4)$$

Proof : We first assume that $c > 0$ and define a function $z(n)$ by the right member of (3), then

$$z(n) - z(n+1) = f(n+1)u(n+1). \qquad \ldots (5)$$

Using the fact that $u(n+1) \leq \sqrt{z(n+1)}$ in (5) we get

$$z(n) - z(n+1) \leq f(n+1)\sqrt{z(n+1)}. \qquad \ldots (6)$$

By using the facts that $\sqrt{z(n+1)} > 0$, $\sqrt{z(n+1)} \leq \sqrt{z(n)}$ for $n \in Z$ and (6), we observe that

$$\begin{aligned} \sqrt{z(n)} - \sqrt{z(n+1)} &= \frac{z(n) - z(n+1)}{\sqrt{z(n)} + \sqrt{z(n+1)}} \\ &\leq \frac{z(n) - z(n+1)}{\sqrt{z(n+1)} + \sqrt{z(n+1)}} \\ &\leq \frac{1}{2} f(n+1). \end{aligned} \qquad \ldots (7)$$

Now set $n = s$ in (7) and sum over $s = n, n+1, ..., m-1$ to obtain the estimate

$$\sqrt{z(n)} - \sqrt{z(m)} \leq \frac{1}{2} \sum_{s=n+1}^{m} f(s) \qquad \ldots (8)$$

Noting that $\lim_{m \to \infty} \sqrt{z(m)} = \sqrt{c}$, the desired inequality in (4) follows by letting $m \to \infty$ in (8) and using the fact that $u(n) \leq \sqrt{z(n)}$ for $n \in Z$.

Now suppose $c = 0$. Then from (3) we see that the inequality

$$u^2(n) \leq \varepsilon + \sum_{s=n+1}^{x} f(s)\, u(s) \qquad \ldots (9)$$

holds for every arbitrary positive number ε, which by the above argument yields the estimate

$$u(n) \leq \sqrt{\varepsilon} + \frac{1}{2} \sum_{s=n+1}^{x} f(s). \qquad \ldots (10)$$

Since $u(n) \geq 0$ and $\varepsilon > 0$ is arbitrary, it follows from (10) that

$$u(n) \leq \frac{1}{2} \sum_{s=n+1}^{x} f(s).$$

This completes the proof of the theorem.

Another useful finite difference inequality is embodied in the following theorem.

Theorem 2 — Let $u(n)$ be a function defined on Z into R_1 and $f(n)$ be a function as defined in Theorem 1 and $c > 1$ be a constant.

$$u(n) \leq c + \sum_{s=n+1}^{x} f(s)\, u(s) \log u(s) \qquad \ldots (11)$$

then

$$u(n) \leq c^{\prod_{s=n+1}^{x} [1 + f(s)]} \qquad \ldots (12)$$

PROOF : Define a function $v(n)$ by the right member of (11), then

$$v(n) - v(n+1) = f(n+1)\, u(n+1) \log u(n+1). \qquad \ldots (13)$$

Using the fact that $u(n+1) \leq v(n+1)$ in (13), we get

$$v(n) \leq [\, 1 + f(n+1) \log v(n+1)\,]\, v(n+1). \qquad \ldots (14)$$

By setting $n = s$ in (14) and then substituting $s = n,\ n+1,\ \ldots,\ m-1$ successively, we get

$$v(n) \leq v(m) \prod_{s=n+1}^{m} [1 + f(s) \log v(s)]. \qquad \ldots (15)$$

Noting that $\lim_{m \to \infty} v(m) = c$ and by letting $m \to \infty$ in (15), we get

$$v(n) \leq c \prod_{s=n+1}^{\infty} [1 + f(s) \log v(s)]$$

$$\leq c \exp \left\{ \sum_{s=n+1}^{\infty} f(s) \log v(s) \right\}. \quad \ldots (16)$$

From (16) we observe that

$$\log v(n) \leq \log c + \sum_{s=n+1}^{\infty} f(s) \log v(s). \quad \ldots (17)$$

Define a function $z(n)$ by the right member of (17), then

$$z(n) - z(n+1) = f(n+1) \log v(n+1). \quad \ldots (18)$$

Using the fact that $\log v(n+1) \leq z(n+1)$ in (18), we get

$$z(n) \leq [1 + f(n+1)]\, z(n+1) \quad \ldots (19)$$

which by following the above argument implies the estimate

$$z(n) \leq \log c \left\{ \prod_{s=n+1}^{\infty} [1 + f(s)] \right\}$$

$$= \log c^{\prod_{s=n+1}^{\infty} [1 + f(s)]} \quad \ldots (20)$$

Using (20) in (17) we have

$$\log v(n) \leq \log c^{\prod_{s=n+1}^{\infty} [1 + f(s)]}$$

which implies

$$v(n) \leq c^{\prod_{s=n+1}^{\infty} [1 + f(s)]} \quad \ldots (21)$$

Now using (21) in (11) we get the required inequality in (12). The proof of the theorem is complete.

The following inequality applies in some situations where Theorems 1 and 2 do not apply.

Theorem 3 — Let $u(n)$, $f(n)$ and $g(n)$ be functions defined on Z into R_+ and $c > 0$ be a constant. If

$$u^2(n) \leq c + \sum_{s=n+1}^{\infty} f(s)\, u(s) \left[u(s) + \sum_{t=s+1}^{\infty} g(t)\, u(t) \right] \quad \ldots (22)$$

$$u(n) \leq \sqrt{c} \left[1 + \frac{1}{2} \sum_{s=n+1}^{\infty} f(s) \prod_{t=s+1}^{\infty} \left[1 + \frac{1}{2} f(t) + g(t) \right] \right]. \quad \ldots (23)$$

PROOF : We first assume that $c > 0$ and define a function $z(n)$ by the right member of (22), them

$$z(n) - z(n+1) = f(n+1)\, u(n+1) \left[u(n+1) + \sum_{t=n+2}^{\infty} g(t)\, u(t) \right]. \quad \ldots (24)$$

Using the fact that $u(n) \leq \sqrt{z(n)}$ for $n \in Z$ in (24), we get

$$z(n) - z(n+1) \leq f(n+1) \sqrt{z(n+1)} \left[\sqrt{z(n+1)} + \sum_{t=n+2}^{\infty} g(t) \sqrt{z(t)} \right]. \quad \ldots (25)$$

Now by following the arguments as in the proof of the inequality (7) in Theorem 1, we get

$$\sqrt{z(n)} - \sqrt{z(n+1)} \leq \frac{1}{2} f(n+1) \left[\sqrt{z(n+1)} + \sum_{t=n+2}^{\infty} g(t) \sqrt{z(t)} \right]. \quad \ldots (26)$$

Now set $n = s$ in (26) and sum over $s = n, n+1, \ldots, m-1$ to obtain the estimate

$$\sqrt{z(n)} \leq \sqrt{z(m)} + \frac{1}{2} \sum_{s=n+1}^{m} f(s) \left[\sqrt{z(s)} + \sum_{t=s+1}^{\infty} g(t) \sqrt{z(t)} \right]. \quad \ldots (27)$$

Noting that $\lim_{m \to \infty} \sqrt{z(m)} = \sqrt{c}$ and by letting $m \to \infty$ in (27), we get

$$\sqrt{z(n)} \leq \sqrt{c} + \frac{1}{2} \sum_{s=n+1}^{m} f(s) \left[\sqrt{z(s)} + \sum_{t=s+1}^{\infty} g(t) \sqrt{z(t)} \right]. \quad \ldots (28)$$

Define a function $v(n)$ by the right member of (28), then

$$v(n) - v(n+1) = \frac{1}{2} f(n+1) \left[\sqrt{z(n+1)} + \sum_{t=n+2}^{\infty} g(t) \sqrt{z(t)} \right]. \quad \ldots (29)$$

Using the fact that $\sqrt{z(n)} \leq v(n)$ for $n \in Z$ in (29), we get

$$v(n) - v(n+1) \leq \frac{1}{2} f(n+1) \left[v(n+1) + \sum_{t=n+2}^{\infty} g(t)\, v(t) \right]. \quad \ldots (30)$$

Define a function $w(n)$ by

$$w(n) = v(n+1) + \sum_{t=n+2}^{\infty} g(t)\, v(t) \quad \ldots (31)$$

then

$$w(n) - w(n+1) = v(n+1) - v(n+2) + g(n+2)\, v(n+2). \quad \ldots (32)$$

Now by making use of (30) and the fact that $v(n+2) \leq w(n+1)$ from (31) in (32), we get

$$w(n) \leq \left[1 + \frac{1}{2} f(n+2) + g(n+2) \right] w(n+1) \quad \ldots (33)$$

which by following the arguments in the proof of Theorem 2 given above implies the estimate

$$w(n) \leq \sqrt{c} \prod_{t=n+2}^{\infty} \left[1 + \frac{1}{2} f(t) + g(t) \right] \quad \ldots (34)$$

Now using (34) in (30), we have

$$v(n) - v(n+1) \leq \frac{1}{2} \sqrt{c}\, f(n+1) \prod_{t=n+2}^{\infty} \left[1 + \frac{1}{2} f(t) + g(t) \right]. \quad \ldots (35)$$

Now set $n = s$ in (35) and sum over $s = n, n+1, \ldots, m-1$ to get the estimate

$$v(n) \leq v(m) + \frac{1}{2} \sqrt{c} \sum_{s=n+1}^{m} f(s) \prod_{t=s+1}^{\infty} \left[1 + \frac{1}{2} f(t) + g(t) \right]. \quad \ldots (36)$$

Noting that $\lim_{m \to \infty} v(m) = \sqrt{c}$, the desired inequality in (23) follows by letting $m \to \infty$ in (36) and using the facts that $u(n) \leq \sqrt{z(n)} \leq v(n)$ for $n \in z$. The proof of the case when $c = 0$ can be completed by following the arguments as in the proof of Theorem 1 given above with suitable changes and hence the proof of inequality (23) is complete.

3. Further Inequalities

In this section, we generalize Theorems 1-3 to functions of two variables.

Theorem 4 — Let $u(m, n)$ and $f(m, n)$ be functions defined on $Z \times Z$ into R_+ and $c \geq 0$ be a constant. If

$$u^2(m, n) \leq c + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f(s, t)\, u(s, t) \qquad \ldots (37)$$

then

$$u(m, n) \leq \sqrt{c} + \frac{1}{2} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f(s, t). \qquad \ldots (38)$$

PROOF : We first assume that $c > 0$ and define a function $z(m, n)$ by the right member of (37), then

$$\begin{aligned} &[\, z(m, n) - z(m+1, n)\,] - [\, z(m, n+1) - z(m+1, n+1)\,] \\ &= f(m+1, n+1)\, u(m+1), n+1) \\ &\leq f(m+1, n+1)\, \sqrt{z(m+1, n+1)}. \end{aligned} \qquad \ldots (39)$$

Here we have used the fact that $u(m+1, n+1) \leq \sqrt{z(m+1, n+1)}$ to get (39). By using

$$\sqrt{z(m, n)} > 0,\ \sqrt{z(m, n+1)} \leq \sqrt{z(m, n)},\ \sqrt{z(m+1, n+1)} \leq \sqrt{z(m+1, n)},$$

$$\sqrt{z(m+1, n+1)} \leq \sqrt{z(m, n+1)} \text{ for } m, n \in z, \text{ we observe that}$$

$$[\, \sqrt{z(m, n)} - \sqrt{z(m+1, n)}\,] = \frac{[\, z(m, n) - z(m+1, n)\,]}{[\, \sqrt{z(m, n)} + \sqrt{z(m+1, n)}\,]} \qquad \ldots (40)$$

$$\begin{aligned} &[\, \sqrt{z(m, n)} - \sqrt{z(m+1, n)}\,] - [\, \sqrt{z(m, n+1)} - \sqrt{z(m+1, n+1)}\,] \\ &= \frac{[\, z(m, n) - z(m+1, n)\,]}{[\, \sqrt{z(m, n)} + \sqrt{z(m+1, n)}\,]} \\ &- \frac{[\, z(m, n+1) - z(m+1, n+1)\,]}{[\, \sqrt{z(m, n+1)} + \sqrt{z(m+1, n+1)}\,]} \\ &\leq \frac{[\, z(m, n) - z(m+1, n)\,]}{[\, \sqrt{z(m, n+1)} + \sqrt{z(m+1, n+1)}\,]} \\ &- \frac{[\, z(m, n+1) - z(m+1, n+1)\,]}{[\, \sqrt{z(m, n+1)} + \sqrt{z(m+1, n+1)}\,]} \\ &= \frac{[\, z(m, n) - z(m+1, n)\,] - [\, z(m, n+1) - z(m+1, n+1)\,]}{[\, \sqrt{z(m, n+1)} + \sqrt{z(m+1, n+1)}\,]} \\ &\leq \frac{[\, z(m, n) - z(m+1, n)\,] - [\, z(m, n+1) - z(m+1, n+1)\,]}{[\, \sqrt{z(m+1, n+1)} + \sqrt{z(m+1, n+1)}\,]} \end{aligned} \qquad \ldots (41)$$

By using (39) in (41) we have

$$[\sqrt{z(m, n)} - \sqrt{z(m + 1, n)}] - [\sqrt{z(m, n + 1)} - \sqrt{z(m + 1, n + 1)}]$$

$$\leq \frac{1}{2} f(m + 1, n + 1). \qquad \ldots (42)$$

Now keeping m fixed in (42), set $n = t$ and sum over $t = n, n + 1, \ldots, q - 1$ to obtain the estimate

$$[\sqrt{z(m, n)} - \sqrt{z(m + 1, n)}] - [\sqrt{z(m, q)} - \sqrt{z(m + 1, q)}]$$

$$\leq \frac{1}{2} \sum_{t=n+1}^{q} f(m + 1, t). \qquad \ldots (43)$$

Noting that $\lim_{q \to \infty} \sqrt{z(m, q)} = \lim_{q \to \infty} \sqrt{z(m + 1, q)} = \sqrt{c}$, and by letting $q \to \infty$ in (43) we get

$$\sqrt{z(m, n)} - \sqrt{z(m + 1, n)} \leq \frac{1}{2} \sum_{t=n+1}^{\infty} f(m + 1, t). \qquad \ldots (44)$$

Keeping n fixed in (44), set $m = s$ and sum over $s = m, m + 1, \ldots, p - 1$ to obtain the estimate

$$\sqrt{z(m, n)} - \sqrt{z(p, n)} \leq \frac{1}{2} \sum_{s=m+1}^{p} \sum_{t=n+1}^{\infty} f(s, t). \qquad \ldots (45)$$

Noting that $\lim_{p \to \infty} \sqrt{z(p, n)} = \sqrt{c}$, the required inequality in (38) follows by letting $p \to \infty$ in (45) and using the fact that $u(m, n) \leq \sqrt{z(m, n)}$ for $m, n \in Z$. The proof of the case when $c = 0$ can be completed as in the proof of Theorem 1.

Theorem 5 — Let $u(m, n)$ be a function defined on $Z \times Z$ into R_1 and $f(m, n)$ be a function as defined in Theorem 4 and $c \geq 1$ be a constant. If

$$u(m, n) \leq c + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f(s, t)\, u(s, t) \log (s, t) \qquad \ldots (46)$$

then

$$u(m, n) \leq c^{\prod_{s=m+1}^{\infty} \left| 1 + \sum_{t=n+1}^{\infty} f(s, t) \right|} \qquad \ldots (47)$$

PROOF : Define a function $z(m, n)$ by the right member of (46), then

$$[z(m, n) - z(m + 1, n)] - [z(m, n + 1) - z(m + 1, n + 1)]$$

$$= f(m + 1, n + 1)\, u(m + 1, n + 1) \log u(m + 1, n + 1)$$

$$\leq f(m + 1, n + 1)\, z(m + 1, n + 1) \log z(m + 1, n + 1). \quad \ldots (48)$$

Here we have used the fact that $u(m + 1, n + 1) \leq z(m + 1, n + 1)$ to get the last inequality in (48). From the definition of $z(m, n)$ we see that $z(m + 1, n + 1) \leq z(m + 1, n)$ for $m, n \in z$. Using this fact we observe from (48) that

$$\frac{[z(m, n) - z(m + 1, n)]}{z(m + 1, n)} - \frac{[z(m, n + 1) - z(m + 1, n + 1)]}{z(m + 1, n + 1)}$$

$$\leq f(m + 1, n + 1) \log z(m + 1, n + 1). \quad \ldots (49)$$

Now keeping m fixed in (49), set $n = t$ and sum over $t = n, n + 1, \ldots, q - 1$ to obtain the estimate

$$\frac{[z(m, n) - z(m + 1, n)]}{z(m + 1, n)} - \frac{[z(m, q) - z(m + 1, q)]}{z(m + 1, q)}$$

$$\leq \sum_{t=n+1}^{q} f(m + 1, t) \log z(m + 1, t). \quad \ldots (50)$$

Noting that $\lim_{q \to \infty} z(m, q) = \lim_{q \to \infty} z(m + 1, q) = c$ and by letting $q \to \infty$ in (50) we get

$$z(m, n) \leq \left[1 + \sum_{t=n+1}^{\infty} f(m + 1, t) \log z(m + 1, t)\right] z(m + 1, n). \quad \ldots (51)$$

By keeping n fixed in (51) and setting $m = s$ and then substituting $s = m, m + 1, \ldots, p - 1$ successively, we obtain the estimate

$$z(m, n) \leq z(p, n) \prod_{s=m+1}^{p} \left[1 + \sum_{t=n+1}^{\infty} f(s, t) \log z(s, t)\right]. \quad \ldots (52)$$

Noting that $\lim_{p \to \infty} z(p, n) = c$ and by letting $p \to \infty$ in (52) we get

$$z(m, n) \leq c \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} f(s, t) \log z(s, t)\right].$$

$$\leq c \exp\left\{\sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f(s, t) \log z(s, t)\right\}. \quad \ldots (53)$$

From (53) we observe that

$$\log z(m, n) \leq \log c + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f(s, t) \log z(s, t). \quad \ldots (54)$$

From (54) and by following exactly the same arguments as above with suitable changes up to the inequality (53), we obtain the estimate

$$\log z(m, n) \leq \log c \left\{ \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} f(s, t) \right] \right\}$$

$$= \log c \left\{ \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} f(s, t) \right] \right\} \quad \ldots (55)$$

From (55) we observe that

$$z(m, n) \leq c \left\{ \prod_{s=m+1}^{\infty} \left[1 + \sum_{t=n+1}^{\infty} f(s, t) \right] \right\} \quad \ldots (56)$$

Using (56) in (46) we get the required inequality in (47) and the proof of the theorem is complete.

Theorem 6 — Let $u(m, n)$, $f(m, n)$ and $g(m, n)$ be functions defined on $Z \times Z$ into R_+ and $c \geq 0$ be a constant. If

$$u^2(m, n) \leq c + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f(s, t)\, u(s, t) \left[u(s, t) + \sum_{k=s+1}^{\infty} \sum_{r=t+1}^{\infty} g(k, r)\, u(k, r) \right]. \quad \ldots (57)$$

then

$$u(m, n) \leq \sqrt{c} \left[1 + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \frac{1}{2} f(s, t) \cdot \prod_{k=s+1}^{\infty} \left[1 + \sum_{r=t+1}^{\infty} \left[\frac{1}{2} f(k, r) + g(k, r) \right] \right] \right] \quad \ldots (58)$$

The proof of this theorem is similar to the proofs of Theorems 3-5 given above

with suitable modifications and hence we omit the details

In concluding this section we note that the inequalities established in Theorems 4-6 can be extended very easily to functions of more than two variables.

4. Some Applications

In this section we indicate some applications of our results to obtain bounds on the solutions of certain sum-difference equations for which the earlier inequalities do not apply directly. For example, consider the following sum-difference equation

$$u^2(m, n) = a(m, n) + \sum_{s=m+1}^{x} \sum_{t=n+1}^{x} F[m, n, s, t, u(s, t)]. \quad \ldots (59)$$

for $m, n, \in Z$, where $a(m, n)$ is a function defined for $m, n, \in Z$ into R and $F[m, n, s, t, u]$ is a function defined for into R $m, n, s, t, \in z$, R into R. We assume that

$$|a(m, n)| \leq c, \ |F[m, n, s, t, u(s, t)]| \leq f(s, t)|u(s, t)|, \quad \ldots (60)$$

where $c \geq 0$ is a constant and $f(m, n)$ is a function defined for $m, n \in Z$ into R_+. From (59) and (60) we obtain

$$|u(m, n)|^2 \leq c + \sum_{s=m+1}^{x} \sum_{t=n+1}^{x} f(s, t)|u(s, t)|. \quad \ldots (61)$$

Now an application of Theorem 4 yields

$$|u(m, n)| \leq \sqrt{c} + \frac{1}{2} \sum_{s=m+1}^{x} \sum_{t=n+1}^{x} f(s, t). \quad \ldots (62)$$

The right side in (62) gives us a bound on the solution $u(m, n)$ of the equation (59).

We finally note that the inequalities given in Theorems 5 and 6 can be used to obtain bounds on the solutions of the following sum-difference equations

$$u(m, n) = a(m, n) + \sum_{s=m+1}^{x} \sum_{t=n+1}^{x} u(s, t) F[s, t, \log u(s, t)] \quad \ldots (63)$$

and

$$u^2(m, n) = a(m, n) + \sum_{s=m+1}^{x} \sum_{t=n+1}^{x} F[m, n, s, t, u(s, t), Tu(s, t)], \quad \ldots (64)$$

where

$$Tu(s, t) = \sum_{k=s+1}^{\infty} \sum_{r=t+1}^{\infty} G(s, t, k, r, u(k, r)) \quad \ldots (65)$$

under some suitable conditions on the function involved in (63) and (64). Various other applications of these inequalities will be reported elsewhere.

REFERENCES

1. F.S. DeBlasi and J. Schinas, *An. Sti. Univ "Al. I. Cuza" Iasi" **20*** (1974), 65-80.
2. A. Mate and P. Nevai, *J. Diff. Eqn.* **53** (1984), 234-57.
3. B. G. Pachpatte, *Proc. Nat. acad. Sci. India* **43** (A) (1973), 348-56.
4. B. G. Pachpatte, *Univ. Beograd Publ. Elek. Fak. Ser. Mat. Fiz.* No. 577-No. 598 (1977), 65-73.
5. B. G. Pachpatte, *Tamkang J. Math.* **12** (1981), 21-33.
6. B. G. Pachpatte, *Indian J. pure appl. Math.* **20** (1989), 1197-1212.
7. C. Papaschinopoulos, *Math. Japonica* **33** (1988), 457-68.
8. G. Papaschinopoulos, *Ann. Soc. Sci.* Bruxelles **102** (1988), 19-28.
9. S. Sugiyama, *Bull. Sci. Engr. Research Lab. Waseda Univ.* **45** (1969), 140-44.
10. D. Willett and J.S.W. Wong, *Monatsh. Math* **69** (1964), 362-67.