Indian J. pure appl. Math., 24(7 & 8) : 455-460, July-August 1993

A CHARACTERIZATION FOR ALPHA CONVEX FUNCTIONS

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(*Received* 3 *July* 1992; *after revision* 5 *January* 1993; *accepted* 5 *March* 1993)

Recently, Burdick, Keogh and Merkes showed that if f is starlike in $U = \{ Z \mid |Z| < 1\}$ then $\mathrm{Re}\{ f(z)/f(-z)\}^{1/2} > 0$ for $z \in U$. In this paper, this intriguing inequality has been extended to α-convex functions and for a class of phi like functions

1. INTRODUCTION

A function $f(z)$ analytic $f(0) = 0$, $f'(0) = 1$, $\frac{f(z)f'(z)}{z} \neq 0$ in the unit disc $U = \{ z \mid |z| < 1 \}$ is called α-convex function if and only if there exists a real number α such that

$$\mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0.$$

Class of α-convex functions is known as Mocanu class.

Recently, Keogh *et al.*[4] proved that if f is starlike then $\mathrm{Re}\{ f(z)/f(-z)\}^{1/2} > 0$ for $z \in U$. This intriguing inequality led Robertson[6] to establish the following.

Theorem [R] — Let $f(z)$ be analytic in U and normalized by $f(0) = 0$, $f'(0) = 1$. Then f is starlike with r espect to the origin if and only if $F_k(z) = [kf(z)/f(kz)]^{1/2}$; $F_k(0) = 1$, $F_0(z) = [f(z)/z]^{1/2}$ are analytic for $-1 < k < 1$ and $F_k(z) \ll (1 + kz)/(1 + z)$ for $z \in U$.

In this paper we shall obtain a similar characterization for Mocanu class of functions and then extend it to a-b-ϕ-like functions which are defined in section 3. We need the following results.

Lemma 1 — Let ϕ and ψ be convex in U and $f \ll \psi$. Then $\phi * f \ll \phi * \psi$.

Lemma 2 — If f is analytic, $f(0) = 0$, $f'(0) = 1$ and $f(z)f'(z)/z \neq 0$ in U and $\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f(z)}\right)\right\} > 0$ then f is starlike.

The proofs of Lemmas 1 and 2 can be found in Theorem 4.1 of Ruscheweyh and Sheil-Small[8] and Bajpai and Mehrok[2] and Mocanu *et al.*[5] respectively.

2. Main Theorem

Our characterization for α--convex functions in terms of subordination is the following.

Theorem 1 — Let $f(z)$ be analytic in U and normalized by the conditions $f(0) = 0$, $f'(0) = 1$ and $f(z)f'(z)/z \neq 0$ in U. Then we have

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \ll \frac{1-z}{1+z} = G(z) \qquad \ldots (1)$$

if and only if

$$\left[\frac{tf(sz)}{sf(tz)}\right]^{(1-\alpha)/2}\left[\frac{f'(sz)}{f'(tz)}\right]^{\alpha/2} \ll \frac{1+tz}{1+sz} \qquad \ldots (2)$$

for all $|s| \leq 1$, $|t| \leq 1$.

This contains theorem R in particular for $\alpha = 0$, and $s = 1$

Proof : First we shall prove the sufficiency part. Let $t = 1$, $s = k$, $0 < k < 1$ and

$$\psi(z) = (1+k)\left[\left\{\frac{f(kz)}{kf(z)}\right\}^{1-\alpha}\left\{\frac{f'(kz)}{f'(z)}\right\}^{\alpha}\right]^{1/2} - 1.$$

Then $\psi(z)$ is analytic in U and $\psi(z) \ll T(z)$ where

$$T(z) = (1+k)\left\{\frac{1+z}{1+kz}\right\} - 1 = \frac{k+z}{1+kz}.$$

Hence $|\psi(z)| < 1$ for all k, $0 < k < 1$. Let $k = 1-\delta$, $0 < \delta < 1$ in $\psi(z)$ and write

$$[f(z-\delta(z)]^{(1-\alpha)/2}\,[f'(z-\delta z)]^{\alpha/2} = \sum_{n=0}^{\infty} b_n(-\delta z)^n$$

in $|\delta z| < 1 - |z|$. Then actual computation gives us.

$$b_0 = [f(z)]^{(1-\alpha)/2}\,[f'(z)]^{\alpha/2}$$

$$b_1 = -\left(\frac{1-\alpha}{2}\right)[f(z)]^{(1-\alpha)/2-1}\,[f'(z)]^{(2+\alpha)/2} - \frac{\alpha}{2}\,[f(z)]^{(1-\alpha)/2}$$

$$[f'(z)]^{(\alpha-1)/2}\, f''(z).$$

Further, if we write

$$B(z) = [f(z)]^{-(1-\alpha)/2}\,[f'(z)]^{-\alpha/2}\,[f(z-\delta z)]^{(1-\alpha)/2}\,[f'(z-\delta z)]^{\alpha/2}$$

then, we get

$$B(z) = 1 - \frac{1}{2}\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(\frac{zf''(z)}{f'(z)}\right)\right]\delta + O(\delta^2).$$

Hence,

$$B(z)(1-\delta)^{(1-\alpha)/2} = B(z)\left[1 - \frac{1}{2}(1-\alpha)\delta + O(\delta^2)\right]$$

$$= 1 - \frac{1}{2}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)} + \alpha - 1\right]\delta + O(\delta^2)$$

and finally

$$(2-\delta)B(z)(1-\delta)^{(1-\alpha)/2} = 2 - \left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha + \alpha\frac{zf''(z)}{f'(z)}\right]\delta + O(\delta^2).$$

This last equation gives

$$\psi(z) = 1 - \left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha + \alpha\frac{zf''(z)}{f'(z)}\right]\delta + O(\delta^2)$$

But Re $\psi(z) \leq |\psi(z)| < 1$ gives

$$-\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] + \operatorname{Re} O(\delta) \leq 0$$

Letting δ be suitably small to get

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \geq 0$$

Since at $z = 0$, the left-hand side is 1, the above inequality converts into a strict inequality. This completes the proof of the first part of Theorem 1. Now we shall establish the necessity part. Define

$$P(z) = \int_0^z \left[\frac{s}{1-sx} - \frac{t}{1-tx}\right] dx\,; \quad s \neq t. \qquad \ldots (3)$$

Then

$$P(0) = 0, \quad P'(0) \neq 0 \text{ and } \operatorname{Re}\left\{1 + \frac{zP''(z)}{P'(z)}\right\} > 0$$

and hence $P(z)$ is convex and univalent in U. Suppose (1) is true. Then by Lemma 1 we have

$$\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{zf''(z)}{f'(z)} - (1-\alpha)\right] * P(z) \ll (G(z) - 1) * P(z)$$

where (*) indicates the Hadamard product for two functions. But the last subordination relation is equivalent to

$$\int_{tz}^{sz} \left[(1-\alpha)\frac{f'(x)}{f(x)} + \alpha\frac{f''(x)}{f'(x)} - \frac{(1-\alpha)}{x} \right] dx \ll -2\log\left[\frac{1+sz}{1+tz}\right].$$

This gives (2) after an exponentiation of the above subordination relation. Thus the proof of Theorem 1 is complete.

3. A Sub-class of ϕ-Like Functions

Let $G(z)$ be a convex conformed mapping of U, $G(0) = 1$. Let $f(z)$ be analytic in U with $f(0) = f'(0) - 1 = 0$. Let $\phi(w)$ be analytic in $f(U)$, $\phi'(0) = 1$, $\phi'(w) \neq 0$ in $f(U) - \{0\}$. Then f is called a-b-ϕ-like if and only if

$$\mathrm{Re}\ \{ a\ A(f) + b\ B\ (f) \} > 0$$

for a and b real and

$$A(f) = 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}$$

and

$$B(f) = \frac{zf'(z)}{\phi(f(z))}.$$

If $a = 0$ and $b = 1$ then these functions are known as ϕ-like. Brickman[3], see also Ruscheweyh[7] showed that ϕ-like functions are univalent and in Bajpai[1] has shown that a-b-ϕ-like functions are also ϕ-like and univalent. For a-b-ϕ-like functions we have the following.

Theorem 2 — Let

$$G(z) = b\left(\frac{1-z}{1+z}\right)$$

$$aA(f) + bB(f) \ll G;\ \ z \in U \qquad \text{... (4)}$$

$$Q(f(w)) = w\exp\left[\int_0^w \left(\frac{f'(x)}{\phi(f(x))} - \frac{1}{x}\right) dx\right] \qquad \text{... (5)}$$

and

$$F(z) = z\exp\left[\int_0^z \left(\frac{G(x)-b}{x}\right) dx\right] \qquad \text{... (6)}$$

If f is a-b-ϕ-like and (4) is satisfied then we have

$$\left[\frac{sf'(sz)\,\phi(f(tz))}{tf'(tz)\,\phi(f(sz))}\right]^a \left[\frac{tQ(f(sz))}{sQ(f(tz))}\right]^b \ll \frac{tF(sz)}{sF(tz)} \qquad \text{... (7)}$$

conversely, (7) gives that f is a-b-ϕ-like.

PROOF : Let f be a-b-ϕ-like and (4) be true. Define

$$H(z) = z\exp\left[\int_0^z \left(\frac{aA(f)+bB(f)-1}{x}\right)dx\right].$$

Then, we get,

$$\frac{zH'(z)}{H(z)} = aA(f(z)) + bB(f(z)).$$

Then (4) gives

$$\frac{zH'(z)}{H(z)} - b \prec G(z) - b.$$

Further, if

$$P(z) = \int_0^z \left[\frac{s}{1-sx} - \frac{t}{1-tx}\right]dx \ , \quad s \neq t$$

then

$$[\, aA(f(z)) + bB(f(z)) - b\,] * P(z) \prec (G(z) - b) * P(z).$$

This gives

$$\int_{tz}^{sz} \left[\frac{aA(f(x)) + bB(f(x)) - b}{x}\right]dx \prec \int_{tz}^{sz}\left[\frac{G(x)-b}{x}\right]dx = \log\left[\frac{tF(sz)}{sF(tz)}\right].$$

The exponentiation of the above subordination relation gives the required result.

Conversely, if (7) is true then we shall show that f is a-b-ϕ-like function. Let us write

$$\psi(z) = \left[\frac{sf'(sz)\,\phi(f(z))}{f'(z)\,\phi(f(sz))}\right]^{a/2b}\left[\frac{Q(f(sz))}{sQ(f(z))}\right]^{1/2}(1+s) - 1$$

and

$$\left[\frac{F(sz)}{sF(z)}\right]^{1/2} = \exp\left[\frac{1}{2}\int_z^{sz}\frac{b}{x}\left[\frac{1-x}{1+x} - 1\right]dx\right] = \left[\frac{1+z}{1+sz}\right]^b.$$

Now, let $s = 1 - \delta$, $0 < \delta < 1$. Then $\psi(z)$ is analytic and bounded by 1 in U. Let us expand $\psi(z)$ in the powers of δ in $|\,\delta z\,| < 1 - |\,z\,|$. Suppose

$$\psi(z) = b_0 + b_1\delta + b_2\delta^2 + \ldots .$$

Then by equating various powers of δ from both sides we get

$$b_0 = 1$$

$$b_1 = \left[-\left(\frac{a-b}{b}\right) - \frac{a}{b}\frac{zf''(z)}{f'(z)} - \frac{zf'(z)Q'(f(z))}{Q(f(z))} - 1 + \frac{a}{b}\frac{z(\phi(f(z)))'}{\phi(f(z))}\right]$$

Since Re $\psi(z) \leq |\psi(z)| < 1$, we get Re $\{\, b_1 + b_2\delta + \ldots \,\} \leq 0$ which gives Re $\{b_1\} \leq 0$ for δ is arbitrary small. This in turn gives

$\text{Re}\,\{\, aA\,(f(z)) + bB\,(f(z))\,\} > 0$ for $b > 0$. Since $aA\,(f(0)) + bB\,(f(0)) = b > 0$. This completes the proof of the theorem.

ACKNOWLEDGEMENT

The author is thankful to the referee for the valuable suggestions.

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