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# AN EXISTENCE THEOREM FOR A FUNCTIONAL-INTEGRAL EQUATION WITH DEVIATING ARGUMENTS

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We prove an existence theorem for a general functional-integral equation with deviating arguments. The proof is based on the notion of measure of weak noncompactness and the Schauder fixed point theorem.

## 1. INTRODUCTION

Integral equations of various types are the subject of important mathematical investigations and often appear in many applications. Existence of solutions of integral equations has been studied by many authors[6-9, 11,12, 14,15]. Banas and Stopke[5] proved an existence theorem for a differential equation with deviating arguments. Balachandran and Ilamaran[2-4] have proved existence theorems for nonlinear integral equations with deviating arguments.

Measure of weak noncompactness plays a major role for the existence of solutions for integral equations in $L^1[a, b]$. Recently Banas[6, 7] proved an existence theorem for a class of nonlinear convolution integral equations and proved the existence of integrable and monotonic solutions of Hammerstein and Urysohn integral equations using the measure of weak noncompactness. Emmanuele[14] studied the existence of integrable solutions of functional-integral equations. In this paper we prove an existence theorem for functional-integral equations with deviating arguments by using the measure of weak noncompactness.

The notion of a measure of weak compactness was introduced by De Blasi[10]. For a nonempty bounded subset $X$ of a Banach space $E$ he defined the measure of weak noncompactness as

$\beta(X) = \inf \{ r > 0 :$ there exists a weakly compact subset $Y$ of $E$ such that $X \subset Y + B_r$

where $B_r$ is the closed ball in $E$ centred at the origin with radius $r$.

Appell and De Pascale[1] proved a formula for $\beta(X)$ in the space $L^1[a, b]$. It is

given by

$$\beta(X) = \lim_{\epsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left\{ \int_D | x(t) | \, dt : D \subset [a, b], \text{ meas } D \leq \epsilon \right\} \right\} \right\}$$

where the symbol meas$D$ stands for the Lebesque measure of a set $D$.

## 2. Basic Assumptions

Consider the equation

$$x(t) = f\left( t, r \int_0^1 k(t, s)\, g(s, x(\sigma_1(s)), x(\sigma_2(s)), \ldots, x(\sigma_n(s)))\, ds \right) \quad \ldots (1)$$

where $x \in L^1[0, 1]$ and $r \geq 0$.

Assume the following five conditions :

(i) $f : [0, 1] \times R \to R$ satisfies Carathéodory conditions and there exist $h_1 \in L^1[0, 1]$ and $b_1 \geq 0$ such that

$$| f(t, x) | \leq h_1(t) + b_1 | x | \quad t \text{ a.e. in } [0, 1],\ x \in R,$$

(ii) $k : [0, 1] \times [0, 1] \to R$ satisfies Carathéodory conditions and the linear operator $K$ defined by

$$(Kx)(t) = \int_0^1 k(t, s)\, x(s)\, ds$$

maps $L[0, 1]$ into itself.

(iii) $g\ [0, 1] \times R^n \to R$ satisfies Caratheodory conditions and there exist $h_2 \in L^1[0, 1]$ and $b_2 \geq 0$ such that

$$| g(t, x_1, x_2, \ldots, x_n) | \leq h_2(t) + b_2 \sum_{i=1}^{n} | x_i |$$

$t$ a.e. in $[0, 1]$, $x \in R$.

(iv) $\sigma_i : [0, 1] \to [0, 1]$ is nondecreasing and there exists positive constants $a_i$ such that

$$\sigma_i'(t) \geq a_i \quad \text{for } i = 1, 2, \ldots, n.$$

(v) $(rb_1 b_2 \, || K ||)/a < 1$, where $a = \sum_{i=1}^{n} a_i$, $r \geq 0$.

## 3. Existence Theorem

*Theorem* — Under the hypotheses (i) to (v), equation (1) has at least one solution in $L^1[0, 1]$.

PROOF : Define the operator $A$ in the set $L^1[0, 1]$ by the formula

$$(Ax)(t) = f(t, r\int_0^1 k(t, s)\, g(s, x(\sigma_1(s)), x(\sigma_2(s)), \ldots, x(\sigma_n(s)))\, ds) \quad \ldots (2)$$

Take $s = (||h_1|| + rb_1 ||h_2|| ||K||)/|1 - (rb_1 b_2 ||K||/a)|$

Let $x \in B_s$. Then we have

$$\int_0^1 |Ax(t)|\, dt = \int_0^1 |f(t, r\int_0^1 k(t, s)\, g(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s)))\, ds|\, dt$$

$$\leq \int_0^1 \left| h_1(t) + rb_1 \left| \int_0^1 k(t, s)\, g(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s)))\, ds \right| \right| dt$$

$$\leq \int_0^1 \left| h_1(t) + rb_1 ||K|| \int_0^1 |g(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s)))|\, ds \right| dt$$

$$\leq ||h_1|| + rb_1 ||K|| \int_0^1 \left| h_2(s) + b_2 \sum_{i=1}^{n} |x(\sigma_i(s))| \right| ds$$

$$\leq ||h_1|| + rb_1 ||K|| ||h_2|| + rb_1 b_2 ||K|| \sum_{i=1}^{n} \int_0^1 |x(\sigma_i(s))|\, (\sigma_i^1(s)/a_i)\, ds$$

$$\leq ||h_1|| + rb_1 ||K|| ||h_2|| + rb_1 b_2 ||K|| \sum_{i=1}^{n} (1/a_i) \int_0^1 |x(s)|\, ds$$

$$\leq ||h_1|| + rb_1 ||K|| ||h_2|| + (rb_1 b_2 ||K|| ||x||/a)$$

$$\leq ||h_1|| + rb_1 ||K|| ||h_2|| + (rb_1 b_2 ||K|| s/a) = s.$$

From this we observe that $A(B_s) \subset B_s$. Moreover from our assumptions (i), (ii) and (iii) follows that the function $A$ is continuous.

Now we show that $\beta(A(X)) \leq (rb_1 b_2 ||K||/a)\ \beta(X)$ for each subset $X$ of $B_s$.

Define two operators $F$, $G$ on $L^1[0, 1]$ with values in $L^1[0, 1]$ by the formulas:

$$(Fx)(t) = f(t, x(t)), \quad (Gx)(t) = g(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s))), \quad t \in [0, 1].$$

For a measurable subset $D \subset [0, 1]$ we have

$$\int_D |(Fx)(t)|\,dt \le \int_D h_1(t)\,dt + b_1 \int_D |x(t)|\,dt,\ x \in X$$

$$\int_D |(Gx)(t)|\,dt \le \int_D h_2(t)\,dt + b_2 \sum_{i=1}^{n} \int_D |x(\sigma_i(t))|\,dt$$

$$\le \int_D h_2(t)\,dt + b_2 \sum_{i=1}^{n} (1/a_i) \int_D |x(\sigma_i(t))|\,\sigma_i'(t)\,dt$$

$$\le \int_D h_2(t)\,dt + (b_2/a) \int_{\sigma(D)} |x(t)|\,dt.$$

Since $\lim_{m(D)\to 0} \int_D h_1(t)\,dt = \lim_{m(D)\to 0} \int_D h_2(t)\,dt = 0$, we have

$$\beta(F(X)) \le b_1 \beta(X) \text{ and } \beta(G(X)) \le (b_2/a)\beta(X). \qquad \ldots (3)$$

Since $K$ is linear and continuous we have

$$\int_D |(Kx)(t)|\,dt \le ||K|| \int_D |x(t)|\,dt.$$

This implies

$$\beta(K(X)) \le ||K||\,\beta(X). \qquad \ldots (4)$$

From (3) and (4) we get

$$\beta(A(X)) = \beta(FrKG(X)) \le (rb_1 b_2 ||K||/a)\beta(X).$$

Put $p = rb_1 b_2 ||K||/a$. Then $p < 1$ and $\beta(A(X)) \le p\beta(X)$.

Define a sequence $(B_s^n)$ of sets as follows :

$$B_s' = \overline{\text{co}}\, A(B_s)$$

and $B_s^{n+1} = \overline{\text{co}}\, A(B_s^n)$ for $n \in N$.

Then $(B_s^n)$ is a decreasing sequence of nonempty, bounded, closed, convex subsets of $B_s$ that are invariant under $A$.

Applying (4) we get

$$\beta(B_s^{n+1}) \le p^{n+1}\,\beta(B_s^n), \qquad n \in N.$$

Consequently $\lim \beta(B_s^n) = 0$.

This implies[10] that $Y = \bigcap_{i=1}^{n} B_s^n$ is a nonempty, closed, convex and relatively weakly compact subset of $B_s$ that is invariant under $A$. It is easy to prove that the

set $AY$ is relatively compact[13]. Application of Schauder's fixed point theorem for the operator $A$ on the set $Y$ yields a fixed point for $A$. This completes the proof.

*Remark* : We observe that if $f(t, u) = \varphi(t) + u$ we get Hammerstein integral equations, whereas when $g(s, v) = v$ we obtain a functional integral equation, the usefulness of it in applications was pointed out in Banas and Knap[7].

*Example* — Consider the following equation

$$x(t) = t^2 + (2/5) \int_0^1 (t + s) [s^3 + (1/2) \sin x(t/3)] \, ds. \qquad \ldots (5)$$

This is clearly of the form (1) and it satisfies all the conditions (i)-(v) with

$$f(t, x) = t^2 + 2x, \quad k(t, s) = t + s$$

$$g(t, x) = t^3 + (1/2) \sin x, \quad n = 1, \; r = 1/5$$

$$\alpha(t) = t/3, \quad b_1 = 2, \quad b_2 = 1/2, \quad a_1 = 1/3$$

$$h_1(t) = t^2, \quad h_2(t) = t^3$$

and $\| K \| = \sup \{ \| Kz \| : \| z \| \leq 1 \} \leq 3/2$.

Therefore from our existence theorem the equation (5) has a solution in $L^1[0, 1]$.

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