Indian J. pure appl. Math., **24** (10); 581-586, October 1993

ON A GENERALISATION OF k_R SPACES

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(Received 6 *April* 1992; *after revision* 2 *April* 1993; *accepted* 28 *June* 1993*)*

We define a l_R space by replacing the compact set by a Lindelof set in the definition of a k_R space and establish its existence by a number of examples. Then we study the different properties of this new space. We also pose the problem, "Does there exist a l_R space which is not a l-space ?".

§ 1. Generalising the notion of k_R spaces, we introduce the concept of l_R spaces in this paper. Section 2 contains the relevant definitions of l_R spaces along with some examples and non-examples, justifying our definition of l_R spaces. After obtaining a few interesting properties of l_R spaces in section 3, we define a μ-space and study its properties.

§ 2. In the follwing discussion, all spaces are assumed to be Hausdorff.

A space X is called a k_R space if each real valued function on X, whose restriction to every compact set $K \subset X$ is continuous, is continuous on X. Warner[1] used this concept but not under the name k_R space. Michael[2] introduced this name k_R space, probably for the first time.

k_R spaces can be characterised by using the notion of *cf*-spaces due to Levine[3] as follows.

Definition 1 — A space X is called a *cf*-space if compact subsets of X are finite.

Using the Definition 1, Mishra[4] deduced the following property of *cf*-spaces in a *P*-space.

Proposition 1 — A T_1 *P*-space is a *cf*-space. A completely regular pseudocompact *cf*-space is discrete.

In the light of the above remarks, we have the following proposition characterising k_R spaces.

Proposition 2 — Let X be a T_1 *cf*-space. Then X is a k_R space if and only if X is discrete.

PROOF : Since the proof is easy, it is omitted.

Definition 2 — A space X is called an l_R space if each real valued function on X whose restriction to every Lindelof set $L \subset X$ is continuous, is continuous on X.

From the definition, we have the following easy proposition.

Proposition 3 — A Lindelof space is an l_R space. Hence any space whose underlying set is countable is an l_R space.

Example 1 — Every k_R space is an l_R space.

Example 2 — Every k-space[5] is a k_R space and therefore a l_R space.

Example 3 — Let $A_0 = [0, \Omega)$ where Ω is the first uncountable ordinal. Let $\mathcal{J}_0$ be the product topology on $X = A_0 \times A_0$. Let $\mathcal{J} \supset \mathcal{J}_0$ be any topology on X. Then $(X, \mathcal{J})$ is a l_R space.

PROOF : Follows from the fact that X is the union of a family of countable open subspaces of X.

Example 4 (Fortissimo Space[6]) — Let $(X, \mathcal{J})$ be an uncountable space in which all points are defined to be isolated except a distinguished point p and an open neighbourhood of p is defined to be any set containing p whose complement is countable. Then $(X, \mathcal{J})$ is a T_5 P-space and an l_R space but not a k_R space. Further it is not pseudocompact.

PROOF : Obviously the space is T_5 and a P-space. So by Proposition 1, it is a *cf*-space. Clearly it is Lindelof and therefore is an l_R space. Since $(X, \mathcal{J})$ is not discrete, by Proposition 2, it is not a k_R space. It is clearly completely regular, in fact normal[6]. So by Proposition 1 it is not pseducompact.

Example 5 — Let N denote the set of all natural numbers with the discrete topology. Let Σ be the set of all permutations of N. Let $\sigma \in \Sigma$. Then $\sigma : N \to \beta N$ has a unique continuous extension to βN, say σ^*. For a fixed $p \in \beta N - N$, we define $\Sigma_p^* = \{\sigma^* p \mid \sigma \in \Sigma\}$. The set Σ has c-points. (c = Cardinality of the continuum) and so Σ_p^* for a fixed $p \in \beta N - N$ has also c-points. Points of N alone are the isolated points in βN (Gillman and Jerison[7], 6.10). So $\beta N - N$ is not discrete. Every infinite compact subset of $\beta N - N$ has cardinality 2^c (Gillman and Jerison[7], 9.12). So every compact subset of $N \cup \Sigma_p^*$ is an l_R space which is not a k_R space. For example if $T_1 \neq \phi$ be any countable subset of Σ_p^*, then $N \bigcup T_1$ will be of the above type.

Example 6 — The following is a non-l_R space which is completely regular and sequentially compact.

Let w_1, w_2 be least ordinals of cardinality $\aleph_1$, $\aleph_2$ respectively. Let $X = [1, w_1) \times [1, w_2) \bigcup \{(w_1, w_2)\}$. $[1, w_1] \times [1, w_2]$ is compact and so completely regular. Being a subspace 1, X is completely regular. $[1, w_1)$ and $[1, w_2)$ are sequentially compact. So X is sequentially compact. Now $P = (w_1, w_2)$ is an isolated

point of every Lindelof set in X containing P is seen by the following. Let $P \in B$ and B be a Lindelof set such that P is a cluster point of B. Then for every $\alpha \in [1, w_1)$ there exists $(x_\alpha, y_\alpha) \in B - (w_1, w_2)$ such that $x_\alpha > \alpha$. Then as the set y_α has a supremum there exists $\lambda \in [1, w_2)$ such that $y_\alpha \leq \lambda$ for all $\alpha \in [1, w_1)$. Now $[1, w_1) \times [1, \lambda]$ is closed in X. Then $F = [B - (w_1, w_2)] \bigcap [1, w_1) \times [1, \lambda]$ is closed in B and hence a Lindelof subset. Then $\pi_1(F)$ is Lindelof where π_1 is the projection onto $[1, w_1)$. But $\pi_1(F) \supset \{x_\alpha \mid \alpha \in [1, w_1)\}$. Therefore, $\pi_1(F)$ is an unbounded Lindelof set in $[1, w_1)$. ... (1)

Also $[1, w_1)$ is not Lindelof[6]. But each countable subset of $[1, w_1)$ has an upper bound in $[1, w_1)$ (Dugundji[6], p. 54). So each Lindelof set in $[1, w_1)$ is bounded which is a contradiction to (1). Define $f : X \to R$ by $f(P) = 1$ and $f = 0$ elsewhere on X. Then restriction of f to every Lindelof set is continuous but f is not continuous on X.

§ 3. First we evolve certain conditions under which an l_R space is an l-space and then obtain a few properties of l_R spaces.

Definition 3 — A space X is called an l-space if $U \subset X$ is such that $U \bigcap L$ is open in L for each Lindelof subset of X, then U is open.

Proposition 4 — Let $(X, \mathcal{J})$ be a space. Let $\mathcal{L}$ be the family of all Lindelof sets in X and let $l(\mathcal{J})$ be the finest topology on X which agrees with $\mathcal{J}$ on every member $L \in \mathcal{L}$.

Then

(i) $\mathcal{J} \subset l(\mathcal{J})$

(ii) For $U \subset X$, $U \in L(\mathcal{J})$ if and only if $U \bigcap L$ is $\mathcal{J}$-open in L for all $L \in \mathcal{L}$.

(iii) $(X, l(\mathcal{J}))$ is an l-space

(iv) $(X, \mathcal{J})$ is an l-space if and only if $\mathcal{J} = l(\mathcal{J})$.

PROOF : Easy consequences of the definitions and properties of weak-topology[8]

Theorem 1 — Let $(X, \mathcal{J})$ be a completely regular l_R space. Then $(X, \mathcal{J})$ is an l-space if and only if $(X, l(\mathcal{J}))$ is completely regular.

PROOF : $(X, \mathcal{J})$ and $(X, l(\mathcal{J}))$ have the same Lindelof subsets on which the two topologies are identical. Since $(X, \mathcal{J})$ is an l_R space, we have $C(X, \mathcal{J}) = C(X, l(\mathcal{J}))$ where as usual $C(X, \mathcal{J})$ denotes the ring of all real valued continuous functions on $(X, \mathcal{J})$. Since completely regular topologies are completely determined by their rings of real valued continuous functions[7], we have $\mathcal{J} = l(\mathcal{J})$.

Proposition 5 — If every point of a space X has a neighbourhood which is an l_R space under the induced topology, then X is an l_R space.

PROOF : Simple and so omitted.

Proposition 6 — (a) Let $(X, \mathcal{J})$ be a completely regular Hausdorff l_R space. Let

G be any clopen subset of X. Then G with the relative topology is also an l_R space.

(b) Let $(X, \mathcal{J})$ be further zero dimensional (in particular a P-space). Then every open subspace of $(X, \mathcal{J})$ is an l_R space.

PROOF : (a) Let f be a real valued function on G such that $f \mid T$ is continuous for every Lindelof subset T of G. Define the function $\tilde{f}$ on X by $\tilde{f}(x) = f(x)$ for $x \in G$ and $\tilde{f}(x) = 0$ for $x \in X - G$.

Let T_1 be a Lindelof subset of X. Let $T = G \cap T_1$ and $T' = (X - G) \cap T_1$. Then T is a Lindelof subset of G and since $f \mid T$ is continuous it follows that $\tilde{f} \mid T_1$ is continuous (as G is a clopen set). So $\tilde{f}$ is continuous and hence $\tilde{f} \mid G = f$ is continuous. So G is an l_R space.

(b) Follows from Proposition 5.

Proposition 7 — A closed subspace of an l_R space need not be an l_R space.

PROOF : Noble[9] has proved that every completely regular Hausdorff space can be embedded as a closed subspace into a pseudocompact completely regular Hausdorff k_R space. Furhter we note that every k_R space is an l_R space.

§ 4. Here we define a μ-space and find a condition under which it is an l_R space in relation to the underlying space. Then by defining l_R continuous functions we show that the property of being an l_R space is invariant under continuous open surjections.

Definition 4 — A space X is said to be a μ-space if every bounded and closed subset of X is compact (A subset M of a space X is said to be bounded if $f \mid M$ is bounded for all $f \in C(X)$).

Example 7 — Real compact spaces and P-spaces are μ-spaces[7].

Definition 5 — Let g be a real valued function on a space X. If $g \mid L$ is continuous for every Lindelof set L of X then g is said to be l_R continuous.

Note : Let X be a space and μX be the intersection of all subspaces of βX that contain X and are μ-spaces, then μX is a μ space and $X \subset \mu X \subset \nu X$.

Proposition 8 — Let X be an l_R space. Let $\mathcal{B}$ be the family of all bounded subsets of X and let $E(X) = \bigcup \{B^{-\nu X} \mid B \in \mathcal{B}\}$. Then $E(X)$ is also an l_R space.

PROOF : Let f be an l_R continuous function on $E(X)$. Then $f \mid X$ is l_R continuous as every Lindelof set in X is Lindelof in $E(X)$. As X is an l_R space $f \mid X$ is continuous. So f can be extended to the whole of νX and therefore to $E(X)$. Let $f \mid X = g$. Then $g \in C(X)$. Let $h : E(X) \to R$ be the continuous extension of g. Let $x \in E(X) - X$. Then $x \in A^{-\nu X}$ for some $A \in \mathcal{B}$. Now $A^{-\nu X} = A^{-E(X)}$ (Note that if A is bounded then so is $\bar{A}$). Hence $A^{-\nu X}$ is compact as νX is a μ-space. So $A^{-E(X)}$ is compact in $E(X)$ and so $f \mid A^{-E(X)}$ is continuous that is f is continuous at x. So $f(x) = h(x)$. Hence $f = h$. So f is continuous on $E(X)$.

Theorem 2 — If X is an l_R space, then μX is also an l_R space.

PROOF : Let $\mathcal{N}_\alpha$ be a cardinal larger that $2^{|\mu X|}$ where $|\mu X| = \text{Card}(\mu X)$. Let w_α be the first ordinal with cardinality $\mathcal{N}_\alpha$. Define

$$B_1 = E(X), \quad B_2 = E(B_1), \quad B_3 = E(B_2 \cup B_1), \ldots$$

$$B_\sigma = E\left(\bigcup_{\delta < \sigma} B_\delta \right), \text{ where } \sigma \in W(w_\alpha).$$

Claim (1) — If each B_δ is an l_R space, then B_σ is an l_R space. Let $W = \bigcup_{\delta < \sigma} B_\delta$. Let f be an l_R-continuous function of W. Then as in Proposition 8, $f | X$ is continuous. Hence $f | X$ be continuously extended to the whole of X and hence to W. Let it be $g : W \to R$ with $g | X = f | X$. Let $x \in B_{\delta_0}$, $\delta_0 < \sigma$. Now f and g are continuous in B_{δ_0} and $f | X = g | X$. So $f(x) = g(x)$ that is $f = g$. The Claim (1) follows from Proposition 8.

Claim (2) — There exists $\sigma \in W(w_\alpha)$ such that $B_\sigma = \mu(X)$. For if not, $\mu(X) \neq B_\sigma$ for all $\sigma \in W(w_\alpha)$. Choose σ_1 such that $|\sigma_1| > |\mu X|$. Let $r < \sigma_1$. Consider $\bigcup_{\delta < r} B_\delta$ and choose a point $x_r \in B_r - \bigcup_{\delta < r} B_\delta$ for every $r < \sigma_1$. Therefore $|B_{\sigma_1} - X| > |\sigma_1|$ since $|\sigma_1| = |\{x_r \mid r \in W(\sigma_1)\}|$ and $x_r \in B_{\sigma_1} - X$ for all $r < \sigma_1$. But $|\sigma_1| > |\mu X| \geq |\mu X - X| \geq |B_{\sigma_1} - X|$ which is a contradiction to the above step.

Proposition 9 — (a) The property of being an l_R space is invariant under continuous surjections which are quotient maps.

(b) In particular it is invariant under continuous open (or closed) surjections.

PROOF : (a) Let $f : X \to Y$ be a continuous surjection which is a quotient map, where X is an l_R space. Let $g : Y \to R$ be an l_R continuous function. Let L be a Lindelof subset of X. Then $f(L)$ is Lindelof. Then $g | f(L)$ is continuous. So $(g | f(L)) \circ (f | L)$ is a continuous map on L, which is evidently $(g \circ f) | L$. So $(g \circ f)$ is continuous on X. Let V be an open set in R. Then $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. Therefore $g^{-1}(V)$ is open in Y.

(b) follows from (a).

In conclusion we wish to point out that we are unable to settle the problem whether there exists a l_R space which is not a l-space.

We thank the referees immensely for their kind and valuable suggestions which helped us to redraft our original paper in the present form.

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