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CESARI INTEGRAL INCLUDES HENSTOCK INTEGRAL

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(Dedicated to the memory of Professor Lamberto Cesari)

We show that Henstock integral in the real interval $[a, b]$ can be formulated under the settings of Cesari integral.

1. BURKILL-CESARI (or CESARI) INTEGRAL

Lamberto Cesari introduced a process of Burkill type integration in two basic papers[1,2] which is remarkable in its generality and applicability and pursued the way of the following abstraction.

Let A be a basic set, $\{I\}$ be a collection of subsets I of $A.T$ stands for a nonempty family of finite system S of sets $I \in \{I\}$, i.e. $S = [I] = [I_1, I_2, \ldots, I_n]$ and S is not necessarily a partition of A. $\mu(S)$, $S \in T$ is a real-valued function defined for all systems $S \in T$. Sets I are sometimes called "intervals" and $\mu(S)$ is called the mesh of S or a mesh function. Concerning the systems $S \in T$, either of the following hypotheses are made. (a) Either (a') A is any set and the sets I of each system $S \in T$ are disjoint; or (a'') A is a topological space, u is the collection of all open sets of A, each set $I \in \{I\}$ possesses interior points and the sets I of each system $S \in T$ are nonoverlapping, i.e. $I, J \in S$ implies

$$I^0, J^0 \neq \varphi, \quad I^0 \bigcap J^0 = I^* \bigcap J^0 = I^0 \bigcap J^* = \varphi$$

where φ is the empty set and 0 and $*$ denote the subsets of the interior and boundary points of a set (in the topology u).

Regarding the mesh function, it is assumed that

$$0 < \mu(S) < +\infty \text{ for every } S \in T$$

and given $\varepsilon > 0$ there are systems $S \in T$ with $0 < \mu(S) < \varepsilon$.

Let $\varphi(I)$ be any real-valued interval function defined for each $I \in \{I\}$. Let

$$\alpha = \liminf_{\mu(S) \to 0} \sum_{I \in S} \varphi(I) \text{ and } \beta = \limsup_{\mu(S) \to 0} \sum_{I \in S} \varphi(I).$$

If $\alpha = \beta$ then the common value is denoted by

$$B = \int_A \varphi \qquad \ldots (1)$$

and is called the Burkill-Cesari (or Cesari) integral of φ with respect to the family T and the mesh function μ.

2. Henstock Integral

Let $f(x)$ be a real-valued function defined on the real interval $[a, b]$. f is said to be Henstock integrable { Henstock[4,5] } to H on $[a, b]$ if for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ defined on $[a, b]$ such that whenever a division D : $[x_{i-1}, x_i]$ given by

$$a = x_0 < x_1 < \ldots < x_n = b \text{ and } \{\xi_1, \xi_2, \ldots, \xi_n\} \qquad \ldots (2)$$

satisfies

$$\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \qquad \ldots (2')$$

for $i = 1, 2, \ldots, n$ we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - H \right| < \varepsilon \qquad \ldots (3)$$

and in this case we write

$$\int_a^b f(x)\, dx = H. \qquad \ldots (4)$$

We show that the integral (4) may be obtained under the settings of the integral (1).

The point ξ_i above is called the associated point of $[x_{i-1}, x_i]$. A division D of $[a, b]$ satisfying the above condition [i.e. (2) and (2')] is said to be δ-fine. The function δ as arises in connection with the integral (4), depends on $\varepsilon > 0$ and as $\varepsilon \to 0+$ we would have to let $\delta(\xi) \to 0+$ at each $\xi > [a, b]$ (Henstock[4], p. 32). It may be noted that given an arbitrary function $\delta(\xi) > 0$ (independent of the notion of integration) in $[a, b]$, there always exists a δ-fine division in $[a, b]$ (Henstock[4], Th. 3.1).

If in the above division D, $x_{i-1} < \xi_i < x_i$ for some i, we can replace $[x_{i-1}, x_i]$ by two intervals $[x_{i-1}, \xi_i]$ and $[\xi_i, x_i]$, so that ξ_i would be at one end point of the interval and this will not change the sum in (3) because

$$f(\xi_i)(x_i - x_{i-1}) = f(\xi_i)(x_i - \xi_i) + f(\xi_i)(\xi_i - x_{i-1})$$

and also clearly the new division remains δ-fine. Therefore we can assume always that the associated points ξ_i are one or other end points of the intervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$ (Henstock[4], p. 33).

3. Theorem

Let $A = [a, b]$ with the usual topology. We have already noted that if $\delta(\xi)$ is any positive function on $[a, b]$ then there always exists a δ-fine division in $[a, b]$. Let such a division be [cf. (2) and (2′)]

$$a = x_0 < x_1 < \ldots < x_n = b \text{ and } \{\xi_1, \xi_2, \ldots, \xi_n\} \qquad \ldots (5)$$

satisfies

$$\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$$

for $i = 1, 2, \ldots, n$. We say that the closed subinterval $[x_{i-1}, x_i]$ belongs to this δ-fine division. There is no harm if one such sub-interval belongs to δ-fine division for more than one δ (i.e. $\delta(\xi)$). As $\delta > 0$ varies, we obtain a family of δ-fine divisions of $[a, b]$, each consists of closed subintervals. Let $\{I\}$ be the collection of all such closed subintervals of $[a, b]$, i.e. if $I \in \{I\}$ then I is of the form $[x_{i-1}, x_i] \subset [a, b]$ and I belongs to δ-fine division for some $\delta > 0$. Let T be the family $\{D\}$ where D is some δ-fine division of $[a, b]$, i.e. if $D \in T$, there exists $\delta > 0$ such that D can be represented as in (2) with (2′). Let

$$\mu(D) = \max_i \; | x_i - x_{i-1} |$$

for $i = 1, 2, \ldots, n$. It is clear that any division $D \in T$ satisfies the condition (a″). Moreover $0 < \mu(D) < +\infty$ and given $\in > 0$ there are divisions $D \in T$ such that $0 < \mu(D) < \varepsilon$.

If $I \in \{I\}$, $I = [c, d]$, we define two interval functions

$$F_l(I) = f(c)(d-c), \quad F_r(I) = f(d)(d-c).$$

We denote the pair (F_l, F_r) by F. If $D : [x_{i-1}, x_i] \in T$, consider the sum

$$\sum_i F(J)$$

where $J = [x_{i-1}, x_i]$ and $F = F_l$ if the associated point $\xi_i = x_{i-1}$ and $F = F_r$ if $\xi_i = x_i$. According to the point noted in the last sentence of Section 2, there is no ambiguity in the definition of the interval function F, because we can now express the division (5) as

$$\{[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]\}$$

where either

$$[x_{i-1}, x_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$$

or $$[x_{i-1}, x_i] \subset (x_{i-1} - \delta(x_{i-1}), x_{i-1} + \delta(x_{i-1}))$$

for $i = 1, 2, ..., n$ and the number of associated points does not exceed the number of subintervals in a division.

Suppose that $f(x)$ is Henstock integrable in $[a, b]$, so that (3) holds and from (1) it follows that

$$H = \int_{[a,b]} F .$$

We therefore obtain the following theorem.

Theorem — Henstock integral in the real interval $[a, b]$ is obtainable as the Cesari integral of a suitable interval function.

Generalisation of the theorem to higher Euclidean space is merely a technical matter.

Note : It is known[1] that Cesari integral includes Riemann, Riemann-Stieltjes, Burkill, Lebesgue, Cauchy, Lebesgue-Stieltjes, Weierstrass integrals and integral on a continuous surface of finite area and Henstock integral includes[5] :

Riemann, Riemann-Stieltjes, Burkill, Lebsegue, Polard-Getchell, Newton, Special Denjoy and Special Denjoy-Stieltjes, Perron and Ward integrals.

It now follows from the theorem that Cesari integral includes also the integrals enlisted above.

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References

1. L. Cesari, *Trans. Am. Math. Soc* **102** (1962), 94-113.
2. L. Cesari, *Trans. Am. Math. Soc.* **102** (1962), 114-46.
3. R. Henstock, *Theory of Integration*, Butterworths, 1963.
4. R. Henstock, *Lectures on the Theory of Integration*, World Scientific Pub. Co. (P) Ltd., 1988.
5. L. Peng-Yee, *Lanzhou Lectures on Henstock Integration*, World Scientific Pub. Co. (P) Ltd., 1989.