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SOME NEW SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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In this paper we introduce and examine some properties of three sequence spaces defined using a modulus function.

1. INTRODUCTION

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional φ on **m**, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if and only if

(1) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(2) $\varphi(e) = 1$ where $e = (1, 1, 1, ...)$, and

(3) $\varphi((x_{\sigma(n)})) = \varphi(x)$ for all $x \in \mathbf{m}$.

For certain kinds of mappings σ, every invariant mean φ extends the limit functional on the space c of real convergent sequences, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose σ-means are equal[8].

When $\sigma(n) = n + 1$, the σ-mean are the classical Banach limits on **m** and V_σ is the set of almost convergent sequences[1].

The mappings σ are assumed one-to-one and such that $\sigma^k(n) \neq n$ for all positive integers n and k, where $\sigma^k(n)$ denotes the kth iterate of the mapping σ at n.

If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown [8] that

$$V_\sigma = \{x = (x_n) : \lim_m t_{mn}(x) = Le, \text{ uniformly in } n, \ L = \sigma\text{-}\lim x\}$$

where

$$t_{mn}(x) = (x_n + Tx_n + T^m x_n)/(m + 1).$$

Recently, Mursaleen[4] defined strongly σ-convergent sequences replacing the Banach limits by σ-means, in the following manner :

A bounded sequence $x = \{x_k\}$ is said to be strongly σ-convergent to a number L if and only if

$$\lim_m 1/m \sum_{k=1}^{m} | x_{\sigma^k(n)} - L | \to 0, \text{ uniformly in } n. \quad (*)$$

By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences. When (*) holds we write $[\sigma]$-lim $x = L$. For $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the space of strongly almost convergent sequences.

If $A = (a_{mk})$ is a nonnegative matrix, Σ_k denotes the summation $k = 1$ to ∞ and w denotes all complex valued sequences, then we define Savas[7],

$$w_\sigma(A_\sigma) = \left\{ x \in w : \lim_m \sum_k a_{mk} \; | x_{\sigma^k(n)} | = 0, \quad \text{uniformly in } n \right\}$$

$$w(A_\sigma) = \left\{ x \in w : \lim_m \sum_k a_{mk} \; | x_{\sigma^k(n)} - L | = 0, \text{ for some } L \text{ uniformly in } n \right\}$$

and

$$w_\infty(A_\sigma) = \left\{ x \in w : \sup_{m,n} \sum_k a_{mk} \; | x_{\sigma^k(n)} | < \infty \right\}.$$

The collection $w(A_\sigma)$ is commonly referred to as the collection of the strongly invariant A-summable sequences.

In the present note we introduce some new sequence spaces by using a modulus function f and examine some properties of these sequence spaces.

2. Main Results

Definition 1[5] — A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

(a) $f(x) = 0$ if and only if $x = 0$,

(b) $f(x + y) \leq f(x) + f(y)$,

(c) f is increasing and

(d) f is continuous from the right at 0.

Definition 2 — Let f be a modulus and $A = (a_{mk})$ be a nonnegative regular matrix. We define

$$w_\sigma(A_\sigma, f) = \left\{ x \in w : \lim_m \sum_k a_{mk} \, f(\, | x_{\sigma^k(n)} | \,) = 0, \quad \text{uniformly in } n \right\}$$

$$w(A_\sigma, f) = \left\{ x \in w : \lim_m \sum_k a_{mk} f(| x_{\sigma^k(n)} - L |) = 0, \text{ for some } L \text{ uniformly in } n \right\}$$

and

$$w_\infty(A_\sigma, f) = \left\{ x \in w : \sup_{m,n} \sum_k a_{mk} f(\mid x_{\sigma^k(n)} \mid) < \infty \right\}.$$

If $x \in w(A_\sigma, f)$ we say that x is strongly invariant A-summable to L with respect to the modulus f.

When $\sigma(n) = n + 1$, we have the following sequences spaces :

$$w_0(\hat{A}, f) = \left\{ x \in w : \lim_m \sum_k a_{mk} f(\mid x_{k+n} \mid) = 0, \quad \text{uniformly in } n \right\}$$

$$w_0(\hat{A}, f) = \left\{ x \in w : \lim_m \sum_k a_{mk} f(\mid x_{k+n} - L \mid) = 0, \text{ for some } L \text{ uniformly in } n \right\}$$

and

$$w_\infty(\hat{A}, f) = \left\{ x \in w : \sup_{m,n} \sum_k a_{mk} f(\mid x_{k+n} \mid) < \infty \right\}.$$

If $x \in w(\hat{A}, f)$, we say that x is strongly almost A-summable to L with respect to the modulus f.

When $A = (a_{mk}) = (C, 1)$ Cesaro matrix, we have the following sequence spaces which are generalization of the sequence spaces $[V_\sigma]_0$, $[V_\sigma]$ and $[V_\sigma]_\infty$ which were defined by Savas[6] :

$$[V_\sigma, f]_0 = \left\{ x \in w : \lim_m \frac{1}{m} \sum_{k=1}^{m} f(\mid x_{k+n} \mid) = 0, \quad \text{uniformly in } n \right\}$$

$$[V_\sigma, f] = \left\{ x \in w : \lim_m \frac{1}{m} \sum_{k=1}^{m} f(\mid x_{k+n} - L \mid) = 0, \text{ for some } L \text{ uniformly in } n \right\}$$

and

$$[V_\sigma, f]_\infty = \left\{ x \in w : \sup_{m,n} \frac{1}{m} \sum_{k=1}^{m} f(\mid x_{k+n} \mid) < \infty \right\}.$$

If $x \in [V_\sigma(f)]$, we say that x is strongly σ-convergent to L with respect to the modulus f.

We now establish a number of useful theorems.

Theorem 1 — $w_0(A_\sigma, f)$, $w(A_\sigma, f)$, and $w_\infty(A_\sigma, f)$ are linear spaces over the complex field C.

PROOF : We consider only $w_0(A_\sigma, f)$. Others can be treated similarly.

Let x, $y \in w_0(A_\sigma, f)$. For $\lambda, \mu \in C$, there exists M_λ and N_μ integers such that $\mid \lambda \mid \leq M_\lambda$ and $\mid \mu \mid \leq N_\mu$. From Definition 1(b), we write

$$\sum_k a_{mk} f(| \lambda x_{\sigma^k(n)} + \mu y_{\sigma^k(n)} |) \le M_\lambda \sum_k a_{mk} f(| x_{\sigma^k(n)} |)$$

$$+ N_\mu \sum_k a_{mk} f(| y_{\sigma^k(n)} |).$$

For $m \to \infty$, since $x, y \in w_0(A_\sigma, f)$, we have $\lambda x + \mu y \in w_0(A_\sigma, f)$. Thus $w_0(A_\sigma, f)$ is linear space over C.

Theorem 2 — Let A be a nonnegative regular matrix and f be a modulus, then $w_\infty(A_\sigma, f) \supset w(A_\sigma, f)$.

PROOF : Let $x \in w(A_\sigma, f)$. Now by Definition 1(b),

$$\sum_k a_{mk} f(| x_{\sigma^k(n)} |) = \sum_k a_{mk} f(| x_{\sigma^k(n)} - L + L |)$$

$$\le \sum_k a_{mk} f(| x_{\sigma^k(n)} - L |) + f(| L |) \sum_k a_{mk}.$$

There exists an integer M_L such that $| L | \le M_L$. Hence we have

$$\sum_k a_{mk} f(| x_{\sigma^k(n)} |) \le \sum_k a_{mk} f(| x_{\sigma^k(n)} - L |) + M_L f(1) \sum_k a_{mk}.$$

Since A is regular and $x \in w(A_\sigma, f)$, we get $x \in w_\infty(A_\sigma, f)$ and this completes the proof.

Theorem 3 — $w_0(A_\sigma, f)$ and $w(A_\sigma, f)$ are complete linear topological spaces paranormed by g defined by

$$g(x) = \sup_{m, n} \sum_k a_{mk} f(| x_{\sigma^k(n)} |).$$

PROOF : From Theorem 2, for each $x \in w(A_\sigma, f)$, $g(x)$ exists. Clearly $g(0) = 0$ and $g(x) = g(-x)$ and $g(x + y) \le g(x) + g(y)$. We now show that the scalar multiplication is continuous.

$$g(\lambda x) = \sup_{m\, n} \sum_k a_{mk} f(| \lambda x_{\sigma^k(n)} |) \le (1 + [| \lambda |]) g(x)$$

where $[| \lambda |]$ denotes the integer part of λ. Whence $\lambda \to 0$, $x \to 0$ imply $g(\lambda x) \to 0$ and also $x \to 0$, λ fixed imply $g(\lambda x) \to 0$. We now show that $\lambda \to 0$, x fixed imply $g(\lambda x) \to 0$.

Let $x \in w(A_\sigma, f)$, then as $m \to \infty$,

$$s_{mn} = \sum_k a_{mk} f(| x_{\sigma^k(n)} - L |) \to 0 \text{ uniformly in } n.$$

For $|\lambda| < 1$, we have

$$\sum_k a_{mk} f(|\lambda x_{\sigma^k(n)}|) = \sum_k a_{mk} f(|\lambda x_{\sigma^k(n)} - \lambda L + \lambda L|)$$

$$\leq \sum_k a_{mk} f(|\lambda x_{\sigma^k(n)} - \lambda L|) + \sum_k a_{mk} f(|\lambda L|)$$

$$\leq \sum_{k>M} a_{mk} f(|\lambda x_{\sigma^k(n)} - \lambda L|) + \sum_{k\leq M} a_{mk} f(|\lambda x_{\sigma^k(n)} - \lambda L|)$$

$$+ \sum_k a_{mk} f(|\lambda L|)$$

$$\leq \sum_{k>M} a_{mk} f(|\leq x_{\sigma^k(n)} - \lambda L|) + \sum_{k\leq M} a_{mk} f(|\lambda x_{\sigma^k(n)} - \lambda L|)$$

$$+ \sum_k a_{mk} f(|\lambda L|).$$

Let $\varepsilon > 0$ and choose M such that for each n, m and $k > M$ implies $s_{nm} < \varepsilon/2$. For each M, by continuity of f, as $\lambda \to 0$,

$$\sum_{k\leq M} a_{mk} f(|\lambda x_{\sigma^k(n)} - \lambda L|) + \sum_k a_{mk} f(|\lambda L|) \to 0.$$

Then choose $\delta < 1$ such that $|\lambda_{,}| < \delta$ implies

$$\sum_{k\leq M} a_{mk} f(|\lambda x_{\sigma^k(n)} - \lambda L|) + \sum_k a_{mk} f(|\lambda L|) < \varepsilon/2$$

Hence we have

$$\sum_k a_{mk} f(|\lambda x_{\sigma^k(n)}|) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and $g(\lambda x) \to 0\ (\lambda \to 0))\}$ Thus $w(A_\sigma, f)$ is paranormed linear topological space by g.

Now, we show that $w(A_\sigma, f)$ is complete with respect to its pranorm topologies. Let $(x^{(s)})$ be a Cauchy sequence in $w(A_\sigma, f)$. Then, we write

$$g(x^{(s)} - x^{(t)}) \to 0, \quad s, t \to \infty$$

i.e., as $s, t \to \infty$, for all n and m, we write.

$$\sum_k a_{mk} f\left(\left| x^{(s)}_{\sigma^k(n)} - x^{(t)}_{\sigma^k(n)} \right|\right) \to 0 \qquad \dots (1)$$

Hence, for each fixed n and k, as $s, t \to \infty$, we have $f\left(\left|\overset{(s)}{x}_{\sigma^k(n)} - \overset{(t)}{x}_{\sigma^k(n)}\right|\right) \to 0$ and for each fixed n and k, $\left(\overset{(s)}{x}_{\sigma^k(n)}\right)_s$ be a Cauchy sequence in C. Since C is complete, as $s \to \infty$, $\left(\overset{(s)}{x}_{\sigma^k(n)}\right)s \to \left(x_{\sigma^k(n)}\right)$ say. Now from (1), we have for $\varepsilon > 0$, there exists a natural number N such that

$$\sum_k a_{mk}\, f\left(\left|\overset{(s)}{x}_{\sigma^k(n)} - \overset{(t)}{x}_{\sigma^k(n)}\right|\right) < \varepsilon \qquad \text{... (2)}$$

for all n, m and $s, t > N$. Since for any fixed natural number M, we have from (2),

$$\sum_{k \le M} a_{mk}\, f\left(\left|\overset{(s)}{x}_{\sigma^k(n)} - \overset{(t)}{x}_{\sigma^k(n)}\right|\right) < \varepsilon \qquad \text{... (3)}$$

for all n, m and $s, t > N$. By taking $t \to \infty$ in the above expression we obtain

$$\sum_{k \le M} a_{mk}\, f\left(\left|\overset{(s)}{x}_{\sigma^k(n)} - x_{\sigma^k(n)}\right|\right) < \varepsilon$$

for all n, m and $s < N$. Since M is arbitrary, by taking $M \to \infty$ we obtain

$$\sum_{k} a_{mk}\, f\left(\left|\overset{(s)}{x}_{\sigma^k(n)} - x_{\sigma^k(n)}\right|\right) < \varepsilon$$

for all n, m and $s > N$ that is, $g(x^{(s)} - x) \to 0$ and $s \to \infty$ and thus $x^s \to x$ as $s \to \infty$.

Also, for each s, there exists $L^{(s)}$ with

$$\sum_k a_{mk}\, f\left(\left|\overset{(s)}{x}_{\sigma^k(n)} - L^{(s)}\right|\right) \to 0 \quad (m \to \infty) \qquad \text{... (4)}$$

uniformly in n. From regularity of A, Definition 1(b) and (4), we have $f(\,|\,L^{(s)} - L^{(t)}\,|\,) \to 0$ $(s, t \to \infty)$ and $(L^{(s)})$ is a Cauchy sequence in C, so $(L^{(s)})$ converges to L, say. Consequently we get

$$\sum_k a_{mk}\, f(\,|\,x_{\sigma^k(n)} - L\,|\,) \to 0 \;\; (m \to \infty)$$

uniformly in n. So that $x \in w(A_\sigma, f)$ and the space is complete.

Using the same technique of Theorem 4 of Maddox[2], it is easy to prove the following theorem.

Theorem 4 — Let A be a nonnegative regular matrix and f be a modulus, then $w_0(A_\sigma, f) \supset w_0(A_\sigma)$, $w(A_\sigma, f) \supset w(A_\sigma)$ and $w_\infty(A_\sigma, f) \supset w_\infty(A_\sigma)$.

Theorem 5 — Let A be a nonnegative regular matrix and f be a modulus. If $\beta = \lim_t (f(t)/t) > 0$ then $w(A_\sigma, f) = w(A_\sigma)$.

PROOF : In Theorem 4, it was shown that $w(A_\sigma, f) \supset w(A_\sigma)$. We must show that $w(A_\sigma) \supset w_0(A_\sigma, f)$. For any modulus function, the existence of positive limit given with β was given in Maddox[3, Proposition 1]. Now $\beta > 0$ and let $x \in w(A_\sigma, f)$. Since $\beta > 0$, for every $t > 0$, we write $f(t) \geq \beta t$. From this inequality, it is easy to see that $x \in w(A_\sigma)$. This completes the proof.

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