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NOTE ON THE ELASTIC WAVE PROPAGATION IN ELASTIC BARS OF FINITE LENGTH WITH RANDOMLY AND LINEARLY VARIYING YOUNG'S MODULUS

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The present problem deals with the investigation of mean and variance of the displacement distribution in a system of elastic bars of finite length with Young's modulus varying slightly linearly and randomly from bar to bar due to time-dependent displacement input at one of the ends, the other ends being kept fixed. Laplace transform on time for small random variations in Young's modulus is utilized and a truncated series solution of the wave problem is achieved. The effects of randomness on the elastic wave motion for three particular cases of different continuous probability distribution of the random variable have been discussed.

1. Introduction

The elastic wave propagation in non-homogeneous media have been investigated by many researchers. The solution for longitudinal propagation of elastic disturbance in a medium for linear variations of elastic parameters was obtained by Datta[2]. Vasudeva and Bhaskara[10] considered a problem on propagation of a pulse in a non-homogeneous elastic rod with varying cross-section. Wave propagation in an elastic non-homogeneous bar of finite length taking into account the variation of the Young's modulus along the bar was discussed by Lindholm and Doshi[4]. All these investigations were deterministic in nature. The problems on wave propagation in continuous media having randomly varying elastic properties have been discussed by some researchers in recent years. Keller[3] have discussed problems on wave motion in random media in geometrical optics, considering small random variations in homogeneity of the medium. Two types of methods – 'honest' and 'dishonest' are utilised in such problems. In the 'honest' method, firstly, a series solution for displacement $u(x, \alpha, t)$ where x is the space co-ordinate, t the time and α a parameter ranging over the probability space S with probability density $p(\alpha)$ is determined for each value of and then its mean value and other statistics such as variance etc. are computed. In the 'dishonest' method, randomness is considered before determining

$u(x, \alpha, t)$ and some assumptions without sufficient reasons are made about the statistical properties of the random wave motion. Following Keller[4], Singh and Singh[8] studied a similar problem in viscoelasticity considering slight random variations in viscoelastic properties. Later, Roychoudhuri and Sain[5] considered a problem on wave propagation in semi-infinite elastic rods of randomly varying cross-section. Recently Roychaudhuri[6] has discussed a problem on wave propagation in semi-infinite elastic bars with randomly varying elastic modulus. In this note the 'honest' method is utilised to obtain a truncated series solution for the mean displacement distribution as well as variance of the displacement distribution on a large sample of elastic bars with Young's modulus varying slightly linearly and randomly from bar to bar due to time-dependent displacement input at one of the ends of the bars, the other ends being kept fixed. Three different continuous probability distributions of the random variable are considered to observe the effect of randomness on the wave motion.

2. Problem Formulation : Governing Equations

Elastic bars of finite length l, cross section A, material density ρ and elastic modulus $E(x, \alpha)$ are considered. A and ρ are constants and are deterministic in nature but the Young's modulus $E(x, \alpha)$ vary slightly and randomly from bar to bar. E is taken as function of x and a parameter α which ranges over a probability space S with probability density $p(\alpha)$. The rods are subjected to a displacement input at one ends $x = l$, the other ends $x = 0$ are kept fixed.

The longitudinal displacement $u(x, t)$ of one such bar is obtained as a solution of the problem[6] :

$$\frac{\partial}{\partial x}\left[AE\,\frac{\partial u}{\partial x}\right] = \rho A\,\frac{\partial^2 u}{\partial t^2} \qquad \ldots(1)$$

$$u(x, 0) = \left.\frac{\partial u}{\partial t}\right]_{t=0} = 0 \qquad \ldots(1a)$$

$$u(x, t) = v_0 H(t) \text{ at } x = l$$

$$= 0 \text{ at } x = 0 \qquad \ldots(1b)$$

where v_0 is a constant and $H(t)$ is the heaviside unit function.

The randomness in Young's modulus is assumed in the form

$$E = E(x, \alpha) = E_0 + \epsilon\, g(x, \alpha) \qquad \ldots(2)$$

where E_0 is constant and $g(x, \alpha)$ is a bounded function of x and α, the parameter α being supposed to be very small in view of the fact that the Young's modulus is of random nature with only slight vartiations.

Substituting eqns. (2) in eqn. (1) and taking Laplace transform on time eqn. (1) raduces to, as in Roychoudhuri[6],

$$\frac{\epsilon}{E_0}\,\frac{dg}{dx}\,\frac{d\bar{u}}{dx} + \left(1 + \frac{\epsilon}{E_0}\,g\right)\frac{d^2\bar{u}}{dx^2} = \frac{s^2}{c^2}\,\bar{u} \qquad \ldots(3)$$

where $c = \sqrt{E_0/\rho}$ is the longitudinal elastic wave velocity in the bar and s the transform variable.

3. Approximate Solution in the Form of a Truncated Series

The solution is assumed in the form,

$$\bar{u}(x, \alpha, s) = \bar{u}_0(x, s) + \frac{\epsilon}{E_0}\bar{u}_1(x, \alpha, s) + \ldots$$

where ϵ/E_0 is very small.

Then as Roychoudhuri[6] the basic equations are

$$\bar{u}_{0,xx} = \frac{s^2}{c^2}\bar{u}_0 \qquad \ldots(4a)$$

$$\bar{u}_{1,xx} + g\bar{u}_{0,xx} + \frac{dg}{dx}\bar{u}_{0,x} = \frac{s^2}{c^2}\bar{u}_1 \qquad \ldots(4b)$$

$$\bar{u}_{n,xx} + g\bar{u}_{n-1,xx} + \frac{dg}{dx}\bar{u}_{n-1,x} = \frac{s^2}{c^2}\bar{u}_n \text{ for } n \geq 1. \qquad \ldots(4c)$$

As in Roychoudhuri[6], the boundary conditions (1b) reduce to

$$\left.\begin{aligned} \bar{u}_0 &= 0 \quad \text{at } x = 0 \\ &= \frac{v_0}{s} \quad \text{at } x = l \end{aligned}\right\} \qquad \ldots(5a)$$

and

$$\left.\begin{aligned} \bar{u}_n &= 0 \quad \text{at } x = 0 \\ &= 0 \quad \text{at } x = l \end{aligned}\right\} \text{ for } n \geq 1. \qquad \ldots(5b)$$

We assume

$$g(x, \alpha) = g_0 \cdot \frac{x}{l} f(\alpha) \qquad \ldots(6)$$

where g_0 is a constant.

The solution of (4a) satisfying the boundary conditions (5a) is given by

$$\bar{u}_0(x, s) = v_0 \frac{\sinh \frac{sx}{c}}{s \sinh \frac{sl}{c}}. \qquad \ldots(7)$$

Inserting the solution for $\bar{u}_0(x, s)$ and (6) into (4b), the solution of (4b) satisfying the conditions (5b) is given by

$$\bar{u}_1(x, \delta, s) = \frac{g_0 \cdot v_0 f(\alpha)}{4}\left[\frac{l}{c}\frac{\cosh sl/c \sinh lx/c}{\sinh^2 sl/c} - \frac{x^2}{cl}\cdot\frac{\cosh sx/c}{\sinh sl/c}\right.$$

$$\left.\left(1 - \frac{x}{l}\right)\cdot\frac{\sinh sx/c}{s\cdot\sinh sl/c}\right]. \qquad \ldots(8)$$

The solution for $\bar{U}_1(x, \alpha, s)$ is then utilized in the next equation to obtain $\bar{U}_2(x, \alpha, s)$ and so on.

$U_0(x, t)$, $U_1(x, \alpha, t)$ etc. for each α are determined by inverting $\bar{U}_0(x, s)$, $\bar{U}_1(x, \alpha, s)$ etc.

Taking inverse Laplace transform of (7) and (8), we obtain (Carslaw and Jagger[8], Thomson[9], Spiegel[10] and the appendix at the end of the paper).

$$\bar{U}_0(x, t) = \varphi_0(x, t) \qquad \ldots(9)$$

where

$$\varphi_0(x, t) = v_0 \left[\left\{ \frac{c}{2l} \left(t + \frac{x}{c} \right) + \frac{1}{\pi} \sum_1^\infty \frac{(-1)^n}{n} \cdot \sin \frac{nc\pi}{l} \left(t + \frac{x}{c} \right) \right\} H\left(t + \frac{x}{c} \right) - \left\{ \frac{c}{2l} \left(t - \frac{x}{c} \right) + \frac{1}{\pi} \sum_1^\infty \frac{(-1)^n}{n} \right.$$

$$\left. \sin \frac{nc\pi}{l} \left(t - \frac{x}{c} \right) \right\} H\left(t - \frac{x}{c} \right) \Bigg\} \qquad \ldots(10)$$

$$U_1(x, \alpha, t) = \varphi_1(x, t) \cdot f(\alpha) \qquad \ldots(11)$$

where

$$\varphi_1(x, t) = \frac{g_0 U_0}{8} \left[\frac{c}{l} \left(t + \frac{x}{c} \right) \left\{ 1 + 2 \sum_1^\infty (-1)^n \cos \frac{nc\pi}{l} \left(t + \frac{x}{c} \right) \right\} \right.$$

$$H\left(t + \frac{x}{c} \right) - \frac{c}{l} \left(t - \frac{x}{c} \right) \left\{ 1 + 2 \sum_1^\infty (-1)^n \cos \frac{nc\pi}{l} \left(t - \frac{x}{c} \right) \right\}$$

$$H\left(t - \frac{x}{c} \right) - \frac{x^2}{l^2} \left\{ 1 + 2 \sum_1^\infty (-1)^n \cos \frac{nc\pi}{l} \left(t + \frac{x}{c} \right) \right\}$$

$$H\left(t + \frac{x}{c} \right) - \frac{x^2}{l^2} \left\{ 1 + 2 \sum_1^\infty (-1)^n \cos \frac{nc\pi}{l} \left(t - \frac{x}{c} \right) \right\}$$

$$H\left(t + \frac{x}{c} \right) + \left(1 - \frac{x}{l} \right) \left\{ \frac{c}{l} \left(t + \frac{x}{c} \right) + \frac{2}{\pi} \sum_1^\infty \frac{(-1)^n}{n} \right.$$

$$\sin \frac{nc\pi}{l} \left(t + \frac{x}{c} \right) \Bigg\} H\left(t + \frac{x}{c} \right) - \left(1 - \frac{x}{l} \right) \left\{ \frac{c}{l} \left(t - \frac{x}{c} \right) \right.$$

$$\left. \left. + \frac{2}{\pi} \sum_1^\infty \frac{(-1)^n}{n} \cdot \sin \frac{nc\pi}{l} \left(t - \frac{x}{c} \right) \right\} H\left(t - \frac{x}{c} \right) \right] \qquad \ldots(12)$$

where

$$H\left(t + \frac{x}{c} \right) = 0, \qquad t \leq -\frac{x}{c}$$

$$= 1 \; t > t > \frac{x}{c}$$

$$H\left(t - \frac{x}{c}\right) = 0 \quad > \quad t \leq \frac{x}{c}$$

$$= 1 \quad > \quad t > \frac{x}{c}.$$

We can thus determine $u_2(x, \alpha, t)$, $u_3(x, \alpha, t)$ and so on.

Assuming the absolute and uniform convergence of the series, the solution for displacement distribution is achieved in the form of truncated series as

$$u(x, \alpha, t) = u_0(x, t) + \frac{\epsilon}{E_0} u_1(x, \delta, t) + \ldots$$

$$\therefore u(x, \alpha, t) = \varphi_0(x, t) + \frac{\epsilon}{E_0} \varphi_1(x, t) f(\alpha) + \ldots\ldots \qquad \ldots(13)$$

where $\varphi_0(x, t)$ and $\varphi_1(x, t)$ are given by (10) and (12) respectively.

4. Computation of Mean and Variance of the Displacement Distribution

Let us assume that the random variable α ranges over the probability space s with probability density $p(\alpha)$, then we can determine the mean displacement distribution in the system of rods given by,

$$\langle u \rangle = \langle u_0 \rangle + \frac{\epsilon}{E_0} \langle u_1 \rangle + \ldots\ldots$$

where, the mean of $f(\alpha)$ is defined by

$$\langle f \rangle \equiv \int f(\alpha)\, p(\alpha)\, d\alpha$$

and var $u = E[u - \langle u \rangle]^2$

where E stands for the expectation of the random variate concerned.

Three Different Probability Distributions

Case (i) – We take the random non-homogeneous function $f(\alpha)$ in the form

$$f(\alpha) = e^{-k\alpha} \qquad \ldots(14)$$

where, k is a positive constant and the distribution of the random variable α is given by the probability density function of gamma type as

$$\left.\begin{aligned} p(\alpha) &= \frac{1}{\Gamma(l_0) \cdot m^{l_0}} \cdot \alpha^{l_0 - 1} e^{-\alpha/m} \quad 0 < \alpha < \infty \\ &= 0 \text{ otherwise} \end{aligned}\right\} \qquad \ldots(15)$$

where l_0 and m are constants and are positive.

$$\therefore \langle f \rangle = \frac{1}{(1 + km)^{l_0}}, \quad \langle f^2 \rangle = \frac{1}{(1 + 2km)^{l_0}}.$$

In this case, the solution for the mean displacement in the form of a truncated series

upto 1st power of ϵ/E_0 is obtained as,

$$\langle u \rangle = \varphi_0(x, t) + \frac{\epsilon}{E_0} \varphi_1(x, t) \frac{1}{(1 + km)^{1/6}}. \quad \ldots(16)$$

Thus, observing (16) we conclude that the effect of the continuous random variable defined by (14) and (15) on the propagation of wave in the system of rods is to multiply the non-homogeneous part of the mean displacement distribution by the constant factor $(1 + km)^{-1/6}$.

Case (ii) – Let

$$f(\alpha) = k\alpha \quad \ldots(17)$$

$$\left.\begin{aligned} p(\alpha) &= 2(1 - \alpha),\ 0 < \alpha < 1 \\ &= 0 \text{ elsewhere.} \end{aligned}\right\} \quad \ldots(18)$$

$$\therefore \langle f \rangle = \frac{k}{3}$$

$$\langle f^2 \rangle = \frac{k^2}{6}$$

$$\therefore \langle u \rangle = \varphi_0(x, t) + \frac{\epsilon}{E_0} \varphi_1(x, t) \cdot \frac{k}{3}. \quad \ldots(19)$$

up to let power of $\frac{\epsilon}{E_0}$.

Thus from (19) we see that due to the random variate represented by (17) and (18), the non-homogeneous part of the mean displacement distribution is multiplied by the constant factor $k/3$.

Case (iii) – Let

$$f(\alpha) = k\alpha \quad \ldots(20)$$

$$\left.\begin{aligned} p(\alpha) &= \frac{1}{2}(\alpha + 1),\ -1 < \alpha < 1 \\ &= 0 \text{ elsewhere.} \end{aligned}\right\} \quad \ldots(21)$$

$$\therefore \langle f \rangle = \frac{k}{3}$$

$$\langle f^2 \rangle = \frac{k^2}{3}$$

$$\therefore \langle u \rangle = \varphi_0(x, t) + \frac{\epsilon}{E_0} \varphi_1(x, t) \cdot \frac{k}{3}. \quad \ldots(22)$$

upto 1st power of $\frac{\epsilon}{E_0}$.

In this case, we conclude that the effect of randomness defined by (20) and (21) on the wave motion is to multiply the non-homogeneous part of the mean displacement distribution by the constant factor $k/3$.

5. Computation of the Variance

We get the approximate solution upto 1st power of ϵ/E_0 as,

$$u(x, \alpha, t) \simeq \varphi_0(x, t) + \frac{\epsilon}{E_0} \varphi_1(x, t) f(\alpha).$$

The determined displacement is a random function for each α and its probability density is also $p(\alpha)$.

$$\text{Var } u = \left(\frac{\epsilon}{E_0}\right)^2 \varphi_1^2 (x, t) \cdot E [f - \langle f \rangle]^2$$

$$= \left(\frac{\epsilon}{E_0}\right)^2 \cdot \varphi_1^2 (x, t) \cdot E [\langle f^2 \rangle - (\langle f \rangle)^2].$$

In case (i),

$$\text{Var } u = \left(\frac{\epsilon}{E_0}\right)^2 \cdot \varphi_1^2 (x, t) \left| \frac{1}{(1+2km)^{l_0}} - \frac{1}{(1+km)^{2l_0}} \right|.$$

In case (ii),

$$\text{Var } u = \left(\frac{\epsilon}{E_0}\right)^2 \cdot \varphi_1^2 (x, t) \cdot \frac{k^2}{18}.$$

In case (iii),

$$\text{Var } u = \left(\frac{\epsilon}{E_0}\right)^2 \cdot \varphi_1^2 (x, t) \cdot \frac{2k}{9}.$$

6. Concluding Remarks

The problem can be solved in the same way as above if instead of the displacement input we prescribe stress-pulse or velocity at the end of the rods.

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