Indian J. pure appl. Math., **24** (11); 691-703, November 1993

FIXED POINT THEOREMS AND STABILITY RESULTS FOR FIXED POINT ITERATION PROCEDURES, II

B. E. RHOADES

Department of Mathematics, Indiana University, Bloomington IN 47405, U.S.A.

(Received 31 *April* 1992; *after revision* 12 *May* 1993; *accepted* 27 *July* 1993*)*

In an earlier paper the author[9] established the stability of several iteration procedures for certain contractive definitions. This paper continues that study by using a more general contractive definition.

Let (X, d) be a complete metric space, T a selfmap of X. Let $x_0 \in X, x_{n+1} := f(T, x_n)$ denote an iteration procedure which yields a sequence of points $\{x_n\}$. Suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\} \subset X$, $\varepsilon_n := d(y_{n+1} f(T, y_n))$. If $\lim \varepsilon_n = 0$ implies that $\lim y_n = p$, then the iteration procedure is said to be T-stable.

The contractive definition we shall use is the following. Suppose there exists a constant c, $0 \leq c < 1$ such that, for each $x, y \in X$,

$$||Tx - Ty|| \leq c \max \{||x - y||, [||x - Tx|| + ||y - Ty||]/2, ||x - Ty||, ||y - Tx||\}. \quad \ldots (1)$$

We shall show that there are several iteration procedures that are T-stable for a mapping T satisfying (1).

Theorem 1 — Let (X, d) be a complete metric space, T a selfmap of X satisfying (1). Let p be the fixed point of T. Let $x_0 \in X$, $x_{n+1} := Tx_n$, $n = 0, 1, \ldots$. Let $\{y_n\} \subset X$ and define $\varepsilon_n := d(y_{n+1}, Ty_n)$, $n = 0, 1, \ldots$. Then

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + m \sum_{i=0}^{n} c^{n+1-i} \, d(x_i, x_{i+1}).$$

$$+ c^{n+1} d(x_0, y_0) + \sum_{i=0}^{n} c^{n-i} \varepsilon_i, \quad \ldots (2)$$

where $m := (1 - c)^{-1}$. Then $\lim y_n = p$ iff $\lim \varepsilon_n = 0$.

PROOF : Since (1) is a special case of the contractive definition of Ciric[5], T has a unique fixed point p and $x_n \to p$.

From (1),

$$d^*(Tx, Ty) \le c \max \{d(x, y), d(x, Tx), [d(x, Tx) + d(x, y) + d(x, Tx) + d(Tx, Ty)]/2, d(x, Tx) + d(Tx, Ty), d(y, x) + d(x, Tx)\}$$

$$\le \max \left\{cd(x, y), cd(x, Tx), \frac{c}{2-c}[2d(x, Tx) + d(x, y)], \frac{c}{1-c} d(x, Tx), cd(x, y) + cd(x, Tx)\right\}$$

$$\le mcd(x, Tx) + cd(x, y). \qquad \ldots (3)$$

We shall now establish (2). Using the triangular inequality and (3),

$$d(y_{n+1}, p) \le d(x_{n+1}, p) + d(x_{n+1}, y_{n+1}) + \varepsilon_n$$

$$= d(x_{n+1}, p) + d(Tx_n, Ty_n) + \varepsilon_n$$

$$\le d(x_{n+1}, p) + mcd(x, Tx_n) + cd(x_n, y_n) + \varepsilon_n.$$

But

$$d(x_n, y_n) \le d(Tx_{n-1}, Ty_{n-1}) + \varepsilon_{n-1}$$

$$\le mcd(x_{n-1}, Tx_{n-1}) + cd(x_{n-1}, y_{n-1}) + \varepsilon_{n+1}.$$

Thus

$$d(y_{n+1}, p) \le d(x_{n+1}, p) + mcd(x_n, Tx_n) + mc^2 d(x_{n-1}, Tx_{n-1}) + c^2 d(x_{n-1}, y_{n-1}) + c\varepsilon_{n-1} + \varepsilon_n.$$

Repeating this process $n - 1$ more times yields (2).

Suppose $\lim \varepsilon_n = 0$. Since $0 \le c < 1$, $\lim c^{n+1} d(x_0, y_0) = 0$.

Since $x_n \to p$, $\lim d(x_{n+1}, p) = 0$. Let A denote the lower triangular matrix with entries $a_{nk} = c^{n-k}$.

Then $\lim_n a_n k = 0$ for each k and $\sum_{k=0}^{n} a_{nk} = (1 - c^{n+1})/(1 - c) \to 1/(1 - c)$ as $n \to \infty$. Therefore A is multiplicative; i.e., for any convergent sequence x, $\lim A_n(x) := \sum_{k=0}^{n} a_{nk} x_k = (1 - c)^{-1} \lim x_n$. Since $\lim \varepsilon_n = 0$,

$$\lim \sum_{k=0}^{n} c^{n-k} \varepsilon_k = 0.$$

Since $\lim d(x_{n+1}, p) = 0$, $\lim d(x_n, x_{n+1}) = 0$. Similarly,

$$\lim cm \sum_{i=0}^{n} c^{n-i} d(x_i, x_{i+1}) = 0.$$

Suppose that $\lim y_n = p$.

$$\begin{aligned}\varepsilon_n &= d(y_{n+1}, Ty_n) \\ &\le d(y_{n+1}, p) + d(p, Ty_n) \\ &\le d(y_{n+1}, p) + mcd(p, Tp) + cd(p, y_n),\end{aligned}$$

and $\lim \varepsilon_n = 0$.

Remark 1 : Theorem 1 is a generalization of both Theorem 2 of Harder and Hicks[7] and Theorem 1 of the author[9].

Theorem 2 — Let $(x, ||\cdot||)$ be a normal linear space, T a selfmap of x satisfying (1). Let p be the fixed point of T. Let $x_0 \in X$ and define $x_{n+1} := (1-\alpha_n)x_n + \alpha_n Tx_n$, where $\{\alpha_n\}$ satisfies (i) $\alpha_0 = 1$, (ii) $0 \le \alpha_n \le 1$ for $n > 0$, (iii) $\sum \alpha_n = \infty$, and

(iv) $\sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_i + c\alpha_i)$ converges.

Let $\{y_n\} \subset X$ and define $\varepsilon_n := ||y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n||$.
Suppose that $\lim x_n = p$. Then

$$\begin{aligned}||p - y_{n+1}|| \le\ & ||p - x_{n+1}|| \\ &+ cm \sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_i + c\alpha_i)\ ||x_j - Tx_j|| \\ &+ \prod_{i=0}^{n} (1 - \alpha_i + c\alpha_i)||x_0 - y_0|| \\ &+ \sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_i + c\alpha_i)\varepsilon_j \qquad \ldots (4)\end{aligned}$$

where it is understood that the product is 1 when $j = n$, and $m = (1-c)^{-1}$. Moreover, $\lim y_n = p$ iff $\lim \varepsilon_n = 0$.

PROOF : We shall first verify (4). Using the triangular inequality and (3),

$$\begin{aligned}||y_{n+1} - p|| &\le ||x_{n+1} - p|| + ||(1-\alpha_n)x_n + \alpha_n Tx_n \\ &\qquad - (1-\alpha_n)y_n - \alpha_n Ty_n|| + \varepsilon_n \\ &\le ||x_{n+1} - p|| + (1-\alpha_n)||x_n - y_n|| + \alpha_n ||Tx_n - Ty_n|| + \varepsilon_n\end{aligned}$$

$$\leq \| x_{n+1} - p \| + (1 - \alpha_n) \| x_n - y_n \| + \alpha_n [mc \| x_n - Tx_n \| + c \| x_n - y_n \|] + \varepsilon_n$$

$$= \| x_{n+1} - p \| + \alpha_n mc \| x_n - Tx_n \| + (1 - \alpha_n + c\alpha_n) \| x_n - y_n \| + \varepsilon_n.$$

But

$$\| x_n - y_n \| \leq \| (1 - \alpha_{n-1}) x_{n-1} + \alpha_{n-1} Tx_{n-1} - (1 - \alpha_n) y_{n-1} - \alpha_{n-1} Ty_{n-1} \| + \varepsilon_{n-1}$$

$$\leq (1 - \alpha_{n-1}) \| x_{n-1} - y_{n-1} \| + \alpha_{n-1} \| Tx_{n-1} - Ty_{n-1} \| + \varepsilon_{n-1}$$

$$\leq (1 - \alpha_{n-1} \| x_{n-1} - y_{n-1} \| + \alpha_{n-1} [mc \| x_{n-1} - Tx_{n-1} \| + c \| x_{n-1} - y_{n-1} \|] + \varepsilon_{n-1}$$

$$= (1 - \alpha_{n-1} + c\alpha_{n-1}) \| x_{n-1} - y_{n-1} \| + mc\, \alpha_{n-1} \| x_{n-1} - Tx_{n-1} \| + \varepsilon_{n-1}.$$

Therefore

$$\| y_{n+1} - p \| \leq \| x_{n+1} - p \| + \alpha_n mc \| x_n - Tx_n \| + (1 - \alpha_n + c\alpha_n) \alpha_{n-1} mc \| x_{n-1} - Tx_{n-1} \| + (1 - \alpha_n + c\alpha_n)(1 - \alpha_{n-1} + c\alpha_{n-1}) \| x_{n-1} - y_{n-1} \| + (1 - \alpha_n + c\alpha_n) \varepsilon_{n-1} + \varepsilon_n.$$

Repeating this process $n - 1$ more times yields (4).

Suppose $\lim \varepsilon_n = 0$. Let A be the lower triangular matrix with entries

$$a_{nj} = \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_j + c\alpha_j).$$

Condition (iii) implies that the product diverges. Hence $\lim_n a_{nj} = 0$ for each j. Condition (iv) implies that the limit of the row sum exists. Therefore A is multiplicative. $\lim x_n = p$ implies that $\lim \| x_n - Tx_n \| = 0$. Since $\lim \varepsilon_n = 0$,

$$\lim cm \sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_j + c\alpha_j) = 0$$

and

$$\lim \sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_j + c\alpha_j) \varepsilon_j = 0.$$

Suppose that $\lim y_n = p$. Then

$$\begin{aligned}\varepsilon_n &= \| y_{n+1} - (1-\alpha_n) y_n \quad \alpha_n T y_n \| \\ &\le \| y_{n+1} - p \| + (1-\alpha_n) \| p - y_n \| + \alpha_n \| p - T y_n \| \\ &\le \| y_{n+1} - p \| + (1-\alpha_n) \| p - y_n \| + mc\alpha_n \| p - Tp \| \\ &\quad + \alpha_n \| p - y_n \|,\end{aligned}$$

and $\lim \varepsilon_n = 0$.

Remark 2 : The author[8] has indicated that $\lim x_n = p$ when x is a Hilbert space, T is a selfmap of C, C a closed convex bounded subset of X. Chidume[1] extended this result to ℓ^p spaces, $p \ge 2$ by adding the assumption of compactness to C. He then extended the result to ℓ^p spaces for $1 < p \le 2$ under the assumption that C is closed convex and bounded and T is continuous[2]. He and Osilike[3] have extended the result to uniformly smooth Bausch spaces with modulus of smoothness of power type $q > 1$, under the hypothesis that C is closed convex and bounded. Chidume and the author[4] have extended the original result to uniformly convex spaces, with C closed and convex. In each case there are additional hypothesis on $\{\alpha_n\}$ other than (i)-(iii).

Remark 3 : Theorem 2 is a generalization of both Theorem 3 of Harder and Hicks[7] and Theorem 2 of the author[9].

Theorem 3 — Let $(X, \| \cdot \|)$ be a normed linear space, T a selfmap of X satisfying (1). Let k be a fixed integer ≥ 1, $x_0 \in X$, $x_{n+1} := \sum_{i=0}^{k} \alpha_i T^i x_n$ $\alpha_i \ge 0$, $\alpha_1 > 0$, $\sum_{i=0}^{k} \alpha_i = 1$. Let p be the fixed point of T. Let $\{y_n\} \subset X$ and $\varepsilon_{n+1} = \| y_{n+1} - \sum_{i=0}^{k} \alpha_i T^i y_n \|$. Then

$$\| p - y_{n+1} \| \le \| p - x_{n+1} \| + \sum_{j=0}^{n} \left[\sum_{i=0}^{k} \alpha_i c^i \right]^j \times \left[\sum_{i=1}^{k} \alpha_i c^i (m+i-1) \right] \| x_{n-j} - T x_{n-j} \| \times \left[\sum_{i=0}^{k} \alpha_i c^i \right]^{n+1} \| x_0 - y_0 \| + \sum_{j=0}^{n} \left[\sum_{i=0}^{k} \alpha_i c^i \right]^{n-j} \varepsilon_j \qquad \ldots (5)$$

where $m = (1-c)^{-1}$. Then $\lim y_n = p$ iff $\lim \varepsilon_n = 0$.

PROOF : We shall first show that $\lim x_n = p$. From (1), $\| Tx_n - Tp \| \le c \| x_n - p \|$. Therefore $\| T^i x_n - T^i p \| \le c^i \| x_n - p \|$.

$$|| x_{n+1} - p || = \left|\left| \sum_{i=0}^{k} \alpha_i T^i x_n - p \right|\right|$$

$$\leq \sum_{i=0}^{k} \alpha_i || T^i x_n - T^i p ||$$

$$\leq \alpha_0 || x_n - p || + \sum_{i=1}^{k} \alpha_i c^i || x_n - p ||$$

$$= \left(\sum_{i=0}^{k} \alpha_i c^i \right) || x_n - p ||$$

$$\leq \left(\sum_{i=0}^{k} \alpha_i c^i \right)^2 || x_{n-1} - p || \ldots \leq \left(\sum_{i=0}^{k} \alpha_i c^i \right)^{n+1} || x_0 - p ||$$

and $|| x_{n+1} - p || = 0$.

We shall now establish (5). From (1),

$$|| Tx - Ty || \leq c \max \{ || x - y ||, [|| x - Tx || + || y - x || + || x - Tx || + || Tx - Ty ||]/2, || x - Tx || + || Tx - Ty ||, || y - x || + || x - Tx ||\},$$

$$\leq \max \left\{ c || x - y ||, \frac{c}{2-c} [2 || x - Tx || + || x - y ||], \frac{c}{1-c} || x - Tx ||, c || x - y || c || x - Tx || \right\}$$

$$= mc || x - Tx || + c || x - y ||.$$

Since $2c/(2 - c) \leq c/(1 - c)$.

$$|| Tx - T^2 x || \leq c \max \{ || x - Tx ||, [|| x - Tx || + || Tx - T^2 x ||]/2, || x - T^2 x ||, 0\}$$

$$= \max \left\{ c || x - Tx ||, \frac{c}{2-c} || x - Tx ||, \frac{c}{1-c} || x - Tx || \right\}$$

$$= cm || x - Tx ||.$$

$$|| T^2 x - T^2 y || \leq c \max \{ || Tx - Ty ||, [|| Tx - T^2 x || + || Ty - T^2 y ||]/2, || Tx - T^2 y ||, || Ty - T^2 x ||\}.$$

$$|| Tx - T^2 y || \leq || Tx - T^2 x || + || T^2 x - T^2 y ||.$$

Thus

$$|| T^2 x - T^2 y || \leq \max \{c || Tx - Ty ||, c^2 [|| x - Tx ||$$

$$+ || y - Ty ||] \frac{c}{1-c}, c || x - Tx ||, c || Ty - Tx ||$$

$$+ c || Tx - T^2 x || \}$$

$$|| x - Tx || + || y - Ty || \leq || x - Tx || + || y - x ||$$

$$+ || x - Tx || + || Tx - Ty ||$$

$$\leq 2 || x - Tx || + || y - x || + mc || x - Tx || + c || x - y ||$$

$$= (2 + mc) || x - Tx || + (c + 1), || x - y ||.$$

Note that $mc + 1 = m$.

$$c || Ty - Tx || + c || Tx - T^2 x || \leq mc^2 || x - Tx ||$$

$$+ c^2 || x - y || + c^2 || x - Tx ||$$

$$= c^2 (m + 1) || x - Tx || 2c^2 || x - y ||.$$

Therefore

$$|| T^2 x - T^2 y || \leq \max \{ mc^2 || x - Tx || + c^2 || x - y ||,$$

$$\frac{c^2 (m+1)}{2} || x - Tx || + \frac{c^2 (c+1)}{2} || x - y ||,$$

$$c^2 (m + 1) || x - Tx || + c^2 || x - y || \}$$

$$= c^2 (m + 1) || x - Tx || + c^2 || x - y ||.$$

In general,

$$|| T^i x - T^i y || \leq c^i (m + i - 1) || x - Tx || + c^i || x - y ||. \qquad \ldots (6)$$

Using the triangular inequality and (6),

$$|| y_{n+1} - p || \leq || x_{n+1} - p || + \left|\left| \sum_{i=0}^{k} \alpha_i T^i x_n - \sum_{i=0}^{k} \alpha_i T^i y_n \right|\right| + \varepsilon_n$$

$$\leq || x_{n+1} - p || + \sum_{i=0}^{k} \alpha_i || T^i x_n - T^i y_n || + \varepsilon_n$$

$$\leq \| x_{n+1} - p \| + \alpha_0 \| x_n - y_n \|$$

$$+ \sum_{i=1}^{k} \alpha_i [c^i (m+i-1) \| x_n - Tx_n \| + c^i \| x_n - y_n \| + \varepsilon_n$$

$$= \| x_{n+1} - p \| + \left(\sum_{i=1}^{k} \alpha_i c^i (m+i-1) \right) \| x_n - Tx_n \|$$

$$+ \left(\sum_{i=0}^{k} \alpha_i c^i \right) \| x_n - y_n \| + \varepsilon_n.$$

But

$$\| x_n - y_n \| \leq \left\| \sum_{i=0}^{k} \alpha_i (T^i x_{n-1} - T^i y_{n-1}) \right\| + \varepsilon_{n-1}$$

$$\leq \sum_{i=0}^{k} \alpha_i \| T^i x_{n-1} - T^i y_{n-1} \| + \varepsilon_{n-1}$$

$$\leq \alpha_0 \| x_{n-1} - y_{n-1} \| + \sum_{i=1}^{k} \alpha_i [c^i (m+i-1)$$

$$\| x_{n-1} - Tx_{n-1} \| + c^i \| x_{n-1} - y_{n-1} \|] + \varepsilon_{n-1}.$$

Thus

$$\| y_{n+1} - p \| \leq \| x_{n+1} - p \| + \sum_{i=1}^{k} \alpha_i c^i (m+i-1)$$

$$\left[\| x_n - Tx_n \| + \left(\sum_{i=0}^{k} \alpha_i c^i \right) \| x_{n-1} - Tx_{n-1} \| \right]$$

$$+ \left(\sum_{i=0}^{k} \alpha_i c^i \right)^2 \| x_{n-1} - y_{n-1} \| + \left(\sum_{i=0}^{k} \alpha_i c^i \right) \varepsilon_{n-1} + \varepsilon_n.$$

Repeating this procedure $n - 1$ more times yi· (5).

Suppose lim $\varepsilon_n = 0$. Define A to be the lower triangular matrix with entries $a_{nj} = \delta^{n-j}$ where

$$\delta := \sum_{i=0}^{k} \alpha_i c^i < 1.$$

Then A is multiplicative and

$$\lim \sum_{j=0}^{n} \delta^{n-j} \varepsilon_j = 0.$$

The other sum can be written in the form

$$\left[\sum_{i=1}^{k} \alpha_i c^i (m+i-1) \right] \sum_{j=0}^{n} \delta^{n-j} \, || x_j - Tx_j ||.$$

Since $\lim x_n = p$, $\lim || x_n - Tx_n || = 0$ and the above sum also has limit zero. Therefore $\lim y_n = p$.

Suppose $\lim y_n = p$. Then

$$\varepsilon_n = \left\| y_{n+1} - \sum_{i=0}^{k} \alpha_i T^i y_n \right\|$$

$$\leq || y_{n+1} - p || + \sum_{i=0}^{k} \alpha_i || T^i p - T^i y_n ||$$

$$\leq || y_{n+1} - p || + \alpha_0 || p - y_n || + \sum_{i=1}^{n} \alpha_i [c^i (m+i-1)$$

$$|| p - Tp || + c^i || y_n - p ||]$$

$$= || y_{n+1} + \left(\sum_{i=0}^{k} \alpha_i c^i \right) || y_n - p ||,$$

and $\lim \varepsilon_n = 0$.

Remark 4 : Theorem 3 is a generalization of both Theorem 4 of Harder and Hicks[7] and Theorem 3 of the author[9].

Our final result involves yet a weaker contractive definition.

Let c be a constant, . Let T $0 \leq c < 1$ satisfy

$$|| Tx - Ty || \leq c \max \{ || x - y ||, [|| x - Tx || + || y - Ty ||]/2 \quad \ldots (7)$$

for all $x, y \in X$.

Theorem 4 — Let $(x, ||\cdot||)$ be a normal linear space, T a selfmap of X satisfying (7). Let $x_0 \in X$, $x_{n+1} := \Sigma_{i=0}^{\infty} \alpha_i T^i x_n$, $\Sigma_{i=0}^{\infty} \alpha_i = 1$ and $\alpha_k \alpha_{k+1} \neq 0$ for at least one integer k. Let p denote the fixed point of T, $\{y_n\} \subset X$ and $\varepsilon_{n+1} := ||y_{n+1} - \Sigma_{i=0}^{\infty} \alpha_i T^i y_n||$. Then

$$||y_{n+1} - p|| \leq ||x_{n+1} - p|| + \delta^{n+1} ||x_0 - y_0|| + \left(2m \sum_{i=1}^{\infty} \alpha_i c^i\right) \sum_{i=0}^{n} \delta^{n-i} ||x_i - Tx_i|| + \sum_{i=0}^{n} \delta^{n-i} \varepsilon_i, \quad \ldots (8)$$

where $m = (2-c)^{-1}$, $\delta := \Sigma_{i=0}^{\infty} \alpha_i c^i$. Then $\lim y_n = p$ iff $\lim \varepsilon_n = 0$.

PROOF : We shall first verify that $\lim x_n = p$.

From (7) $||Tx_n - p|| \leq c ||x_n - p||$.

Therefore $||T^i x_n - p|| \leq c^i ||x_n - p||$.

$$||x_{n+1} - p|| = \left\| \sum_{i=0}^{\infty} \alpha_i T^i x_n - p \right\|$$

$$\leq \sum_{i=0}^{\infty} \alpha_i ||T^i x_n - T^i p||$$

$$\leq \sum_{i=0}^{\infty} \alpha_i c^i ||x_n - p||$$

$$\leq \delta^2 ||x_{n-1} - p|| \leq \ldots \leq \delta^{n+1} ||x_0 - p||$$

and $\lim x_n = p$.

From (7),

$$||Tx - Ty|| \leq c \max \{||x - y||, [||x - Tx|| + ||y - x|| + ||x - Tx|| + ||Tx - Ty||]/2\}$$

$$= \max \left\{ c ||x - y||, \frac{c}{2-c} [2 ||x - Tx|| + ||x - y||] \right\}$$

$$= 2mc ||x - Tx|| + c ||x - y||.$$

$$|| Tx - T^2 x || \le c \max \{ || x - Tx ||, [|| x - Tx || + || Tx - T^2 x ||]/2 \}$$

$$= \max \left\{ c || x - Tx ||, \frac{c}{2-c} || x - Tx || \right\} = c || x - Tx ||.$$

$$|| T^2 x - T^2 y || \le c \max \{ || Tx - Ty ||, [|| Tx - T^2 x || + || Ty - T^2 y ||]/2 \}$$

$$= \max \left\{ c || Tx - Ty ||, \frac{c^2}{2} [|| x - Tx || + || y - Ty ||] \right\}$$

$$|| x - Tx || + || y - Ty || \le || x - Tx || + || y - x ||$$
$$+ || x - Tx || + || Tx - Ty ||$$

$$\le 2 || x - Tx || + || x \cdot y || + 2mc || x - Tx || + c || x - y ||$$

$$= 2(mc + 1) || x - Tx || + (c + 1) || x - y ||$$

Note that $mc + 1 = 2m$. Thus

$$|| T^2 x - T^2 y || \le \max$$

$$\left\{ 2mc^2 || x - Tx || + c^2 || x - y ||, 2mc^2 || x - Tx || + \frac{c^2(c+1)}{2} || x + y || \right\}$$

$$= 2mc^2 || x - Tx || + c^2 || x - y ||.$$

In general,

$$|| T^i x - T^i y || \le 2mc^i || x - Tx || + c^i || x - y ||.$$

We shall now establish (8).

$$|| y_{n+1} - p || \le || x_{n+1} - p || + \left|| \sum_{i=0}^{\infty} \alpha_i T^i x_n - \sum_{i=0}^{\infty} \alpha_i T^i y_n \right|| + \varepsilon_n$$

$$\le || x_{n+1} - p || + \sum_{i=0}^{\infty} || T^i x_n - T^i y_n || + \varepsilon_n$$

$$\le || x_{n+1} - p || + \alpha_0 || x_n - y_n ||$$
$$+ \sum_{i=1}^{\infty} \alpha_i [2mc^i || x_n - Tx_n || + c^i || x_n - y_n ||] + \varepsilon_n$$

$$= || x_{n+1} - p || + \sum_{i=1}^{\infty} \alpha_i 2mc^i || x_n - Tx_n ||$$

$$+ \sum_{i=0}^{\infty} \alpha_i c^i || x_n - y_n ||] + \varepsilon_n.$$

But

$$|| x_n - y_n || \leq \left\| \sum_{i=1}^{\infty} \alpha_i T^i x_{n-1} - \sum_{i=1}^{\infty} \alpha_i T^i y_{n-1} \right\| + \varepsilon_{n-1}$$

$$\leq \sum_{i=0}^{\infty} \alpha_i || T^i x_{n-1} - T^i y_{n-1} || + \varepsilon_{n-1}$$

$$\leq \alpha_0 || x_{n-1} - y_{n-1} || + \sum_{i=1}^{\infty} \alpha_i [2mc^i || x_{n-1} - Tx_{n+1} ||$$

$$+ c^i || x_{n-1} - y_{n-1} ||] + \varepsilon_{n-1}$$

$$= 2m \sum_{i=1}^{\infty} \alpha_i c^i || x_{n-1} - Tx_{n-1} || + \sum_{i=0}^{\infty} \alpha_i c^i$$

$$|| x_{n-1} - y_{n-1} || + c_{n-1}.$$

Thus

$$|| y_{n+1} - p || \leq || x_{n+1} - p || + 2m \sum_{i=1}^{\infty} \alpha_i c^i [|| x_{n-1} - Tx_{n-1} ||$$

$$+ \delta || x_{n-1} - Tx_{n-1} || + \delta^2 || x_{n-1} - y_{n-1} || + \delta \varepsilon_{n-1} + \varepsilon_n.$$

Repeating the process $n - 1$ more times yields (8).

The remainder of the proof is similar to that of the previous theorems and will be omitted.

Remark 5 : Theorem 4 is a generalization of Theorem 5 of the author[9].

Remark 6 : Theorem 4 is extendable to definition (1), provided one adds the hypothesis that $\sum_{i=1}^{\infty} \alpha_i c^i (m^2 + i)$ converges.

Remark 7 : Harder's thesis contains some applications of stability results to first order differential equations.

Remark 8 : In the original version of this paper the author had replaced condition (1) in Theorems 1 and 2 with the following :

$$d(Tx, Ty) \leq c \max \{d(x, y), d(x, Tx), [d(x, Tx) + d(y, Ty)]/2, d(x, Ty), d(y, Tx)\}. \quad \ldots (9)$$

Lj.B. Ciric, in private correspondence, kindly pointed out to the author that (1) and (9) are equivalent.

REFERENCES

1. C. E. Chidume, *J. Nigerian Math. Soc.* **7** (1988), 1-9.
2. C. E. Chidume, *Indian J. pure appl. Math.* **22** (1991), 273-86.
3. C. E. Chidume and M.O. Osilike, president.
4. C. E. Chidume and B.E. Rhoades, preprint.
5. L. Ciric, *Proc. Amer. Math. Soc.* **45** (1974), 267-73.
6. M. Alberta Harder, Ph.D. Dissertation, U. of Missouri-Rolla (1987).
7. M. Alberta Harder and Troy L. Hicks, *Math. Japonica* **33** (1988), 693-706.
8. B. E. Rhoades, *J. Math. Anal. Appl.* **56** (1976), 741-50.
9. B. E. Rhoades, *Indian J. pure appl. Math.* **21** (1990), 1-9.