Indian J. pure appl Math. **24** (12): 757-766, December 1993

ELLIPTIC SURFACES OVER A GENUS 2 CURVE II

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(*Received* 27 *July* 1992; *after revision* 25 *February* 1993; *accepted* 19 *August* 1993)

In this paper we give a classification of minimal elliptic surfaces ($\pi : x \to C$, $g(C) = 2$) with a section and exactly one singular fiber of type I_6^*, provided that the J-map ($J : C \to \mathcal{P}^1$) associated to this fibraction either factors through the canonical map $\varphi_k : C \to \mathcal{P}^1$, or factors through a triple cover of $\mathbf{P}^1$. Also we assume that J^{-1} (1) is either one point of multiplicity 6 or three points each of multiplicity 2.

1. INTRODUCTION

In recent years there have been many aticles[6-8] in which the authors have classified elliptic surfaces over a given curve with a given number of singular fibers and in all of these articles, but one by Stiller[9], the authors have assumed that the base curve is $\mathbf{P}^1$. Stiller[9] assumed that the base curve E is a genus 1 curve. In fact Stiller has classified all minimal elliptic surfaces $P : S \to E$ with a section and exactly one singular fiber of type I_6^*, he proved that there are 4 such surfaces.

In this paper we assume that C is a genus 2 curve, and we classify all minimal elliptic surfaces ($\pi : X \to C$), with a section and exactly one singular fiber of type I_6^*, provided that the J-map factors either through the canonical map, or through a triple cover of $\mathbf{P}^1$.

Remark 1.1 : If $\pi : X \to C$ is a minimal elliptic surface with a section and exactly one singular fiber, then this singular fiber is either of type I_6^* or of type I_{18}^* or of type I_{12}.

Notice that Remark 1.1 is an easy consequence of the following version of Hurwitz's formula (see Miranda and Perison[7]).

$$\sum_{n \geq 1} (n-6)\ (\#\text{ of } I_n\text{-fibers} + \#\text{ of } I_n^*\text{-fibers}) \leq 12\,g(C) - 2$$

$$+\ 4\ (\#\text{ of II-fibers} + \#\text{ of IV}^*\text{-fibers})$$

+ 3 (# of III-fibers + # of III*-fibers)

+ 2 (# of IV-fibers + # of II*-fibers)

where $g(C)$ denotes the genus of the base curve C.

The cases in which the singular fiber is either of type I^*_{18} or of type I_{12}, were treated by the author in another paper which will appear very soon, so in this paper we only concentrate on the case in which the singular fiber is of type I^*_6, but before going into that I will first quote the following results from Section 3 of Miranda[6].

(1.3) To build a minimal elliptic surface $\pi : X \to C$ with a section and a given number of singular fibers, it is enough to build the J-map ($J : C \to \mathbf{P}^1$) associated to this surface

(1.4) The rational minimal elliptic surface $P : S \to \mathbf{P}^1$ whose Weierstrass equation is

$$Y^2 = X^2 - 3t(t-1)^3 x + 2\, t(t-1)^5$$

has $J = t$, and has exactly three singular fibers : a fiber of type II over $t = 0$, a fiber of type III* over $t = 1$, and a fiber to type I_1 over $t = \infty$.

(1.5) If one makes the "transfer of*" process, or the process which "deflates two*" fibres, then the fibers are switched according to the following schedule :

$I_n \leftrightarrow I^*_n \ (n \geq 0)$

II $\leftrightarrow$ IV*

III $\leftrightarrow$ III*

IV $\leftrightarrow$ II*.

Finally a word about notation, the notation $|\,[J^{-1}(x)]\,| = (n_1, \ldots\ldots, n_s)$ will be used to mean that $J^{-1}(x)$ consists of S points say $\{x_1, \ldots., x_s\}$ such that the multiplicity of J at x_i denoted by $m_{x_i}(J)$ is n_i, i.e., $m_{x_i}(J) = n_i$ for $i = 1, 2, \ldots., s$.

Now the plan of this paper goes as follows : in Section 2 we find all possibilities (= 7) for the ramification of the J-map associated to a minimal elliptic surface ($\pi : X \to C$), with a section and exactly one singular fiber of type I^*_6, where C is a genus 2 curve, in Section 3 we construct the J-map ramified as in Table I below, and in Section 4, we build the desired surfaces.

2. Ramification of the J-map

In this section we find all possibilities (=7) for the ramification of the J-map.

Theorem 2.1 — Let C be a genus 2 curve, i.e., $g(C) = 2$, if $\pi : X \to C$ is a minimal elliptic surface with a section and exactly one singular fiber of type I^*_6, then the J-map ($J : C \to \mathcal{P}^1$) associated to this fibration is a degree 6 map whose ramification is given by one of the following seven possibilities :

(2.1.1) $\left| [J^{-1}(\infty)] \right| = 6$, $\left| [J^{-1}(0)] \right| = (3, 3)$ and $\left| [J^{-1}(1)] \right| = (6)$.

(2.1.2) $\left| [J^{-1}(\infty)] \right| = (6)$, $\left| [J^{-1}(0)] \right| = (3, 3)$, $\left| [J^{-1}(1)] \right| = (2, 2, 2)$ and $\left| [J^{-1}(V)] \right| = (3, 1, 1, 1)$ for some $V \in \mathcal{P}^1$, with $v \notin \{0, 1, \infty\}$.

(2.1.3) $\left| [J^{-1}(\infty)] \right| = (6)$, $\left| [J^{-1}(0)] \right| = (3, 3)$, $\left| [J^{-1}(1)] \right| = (2, 2, 2)$ and $\left| [J^{-1}(\eta_i)] \right| = (2, 1, 1, 1, 1)$, $(i = 1, 2)$, with $\eta_1 \neq \eta_2$ in $\mathcal{P}^1$, and $\eta_i \notin \{0, 1, \infty\}$.

(2.1.4) $\left| [J^{-1}(\infty)] \right| = (6)$, $\left| [J^{-1}(0)] \right| = (3, 3)$, $\left| [J^{-1}(1)] \right| = (2, 2, 2)$ and $\left| [J^{-1}(w)] \right| = (2, 2, 1, 1)$ for some $w \in \mathcal{P}^1$, with $w \notin \{0, 1, \infty\}$.

(2.1.5) $\left| [J^{-1}(\infty)] \right| = (6)$, $\left| [J^{-1}(0)] \right| = (6)$, $\left| [J^{-1}(1)] \right| = (2, 2, 2)$ and $\left| [J^{-1}(u)] \right| = (2, 1, 1, 1, 1)$ for some $u \in \mathcal{P}^1$, with $u \notin \{0, 1, \infty\}$.

(2.1.6) $\left| [J^{-1}(\infty)] \right| = (6)$, $\left| [J^{-1}(0)] \right| = (6)$ and $\left| [J^{-1}(1)] \right| = (2, 4)$.

(2.1.7) $\left| [J^{-1}(\infty)] \right| = (6)$, $\left| [J^{-1}(0)] \right| = (3, 3)$, $\left| [J^{-1}(1)] \right| = (2, 4)$ and $\left| [J^{-1}(\lambda)] \right| = (2, 1, 1, 1, 1)$ for some $\lambda \in \mathcal{P}^1$, with $\lambda \notin \{0, 1, \infty\}$.

PROOF : $\mathrm{Deg}(J) = \sum_{n > 1} n$ (# of I_n- fibers + # of I_n^*-fibers) = 6 (see Miranda and Persson[7] 543).

Let R = {ramification points of J}, and let $m_x(J)$ denote the multiplicity of J at x, then by Hurwitz's formula for the genus of a curve we have :

$$2g(C) - 2 = -2\deg(J) + \sum_{x \in R} (m_x(J) - 1), \text{ and hence } \sum_{x \in R} (m_x(J) - 1) = 14.$$

Notice that over ∞ we have the I_6^*-fiber, hence $\sum_{x \in J^{-1}(\infty)} (m_x(J) - 1) = 5$, now if $x \in J^{-1}(0)$ then 3 divides $m_x(J)$, and hence $\left| [J^{-1}(0)] \right| = (3, 3)$ or (6); thus we have two cases to consider

Case A — $\left| [J^{-1}(0)] \right| = (3\ 3)$, hence $\sum_{x \in J^{-1}(\infty)} (m_x(J) - 1) = 4$, i.e.,

$\sum_{x \in R - J^{-1}(0, \infty)} (m_x(J) - 1) = 14 - 9 = 5$. Now if $x \in J^{-1}(1)$, then 2 divides $m_x(J)$, and hence $\left| [J^{-1}(1)] \right| = (2, 2, 2)$ or (6), or $\left| [J^{-1}(1)] \right| = (2, 4)$.

Case A.1 — $\left| [J^{-1}(1)] \right| = 6$, hence $\sum_{x \in J^{-1}(1)} (m_x(J) - 1) = 5$, which gives the right ramification of J; thus in this case the ramification of J is as given in (2.1.1).

Case A.2 — $\left| [J^{-1}(1)] \right| = (2, 2, 2)$, hence $\sum_{x \in J^{-1}(1)} (m_x(J) - 1) = 3$; thus to get the right ramification of J we must have one of the following :

(A.2.1) There is a $V \in \mathcal{P}^1$ with $V \notin \{0, 1, \infty\}$, such that $\left| [J^{-1}(V)] \right| = (3, 1, 1, 1)$, and hence we get the ramification of J is as given in (2.1.2).

(A.2.2) There are η_1 and η_2 in $\mathcal{P}^1$, with $\eta_1 \neq \eta_2$ and $\eta_i \notin \{0, 1, \infty\}$ for $i = 1, 2$, such that $|[J^{-1}(\eta_i)]| = (2, 2, 1, 1)$, and hence we get the ramification of J as given in (2.1.3).

(A.2.3) There is a W in $\mathbf{P}^1$, with $W \notin \{0, 1, \infty\}$, such that $|[J^{-1}(W)]| = (2, 2, 1, 1)$, and hence we get the ramification of J as given in (2.1.4).

Case A.3 — $|[J^{-1}(1)]| = (2, 4)$, hence $\sum_{x \in J^{-1}(1)} (m_x(J) - 1) = 4$, and to get the right ramification of J, there must exist a $\lambda \in \mathcal{P}^1$, with $\lambda \notin \{0\ 1, \infty\}$, such that $|[J^{-1}(\lambda)]| = (2, 1, 1, 1, 1)$, which gives the ramification of J as given above in (2.1.7).

Case B — $|[J^{-1}(0)]| = (6)$; thus $\sum_{x \in J^{-1}(0)} (m_x(J) - 1) = 5$, and to get the right ramification of J we notice that $|[J^{-1}(1)]| \neq 6$, and hence either $|[J^{-1}(1)]| = (2, 4)$, which gives the ramification of J as given in (2.1.6), or $|[J^{-1}(1)]| = (2, 2, 2)$, and hence there is a u in $\mathcal{P}^1$, with $u \notin \{0, 1, \infty\}$ such that $|[J^{-1}(u)]| = (2, 1, 1, 1, 1)$, and hence we get the ramification of J as given in (2.1.5). Q.E.D.

The results of Theorem 2.1 are summarized in Table I below :

TABLE I : Possible ramification of the J-map (J: $C \to \mathcal{P}^1$) described in Theorem 2.1 above, with $|[J^{-1}(\infty)]| = (6)$

#	$\lvert[J^{-1}(0)]\rvert$	$\lvert[J^{-1}(1)]\rvert$	ramification of J over points $\neq 0, 1, \infty$
1	(3, 3)	(6)	
2	(3, 3)	(2, 2, 2)	$\lvert[J^{-1}(V)]\rvert = (3, 1, 1, 1)$
3	(3, 3)	(2, 2, 2)	$\lvert[J^{-1}(\eta_i)]\rvert = (2, 1, 1, 1, 1)$, $i = 1, 2$, $\eta_1 \neq \eta_2$
4	(3, 3)	(2, 2, 2)	$\lvert[J^{-1}(W)]\rvert = (2, 2, 1, 1)$
5	(6)	(2, 2, 2)	$\lvert[J^{-1}(u)]\rvert = (2, 1, 1, 1, 1)$
6	(6)	(2, 4)	
7	(3, 3)	(2. 4)	$\lvert[J^{-1}(\lambda)]\rvert = (2, 1, 1, 1, 1)$

Remark 2.3 : In the rest of this paper we restrict ourselves to the first five possibilities of Table I above.

3. THE J-MAP

In this section we will find the J-map ($J : C \to \mathcal{P}^1$) associated to a minimal elliptic surface $\pi : X \to C$, ($g(C) = 2$), with a section and exactly one fiber of type I_σ^*. In fact we will obtain such a J-map by answering the following two questions :

(3.1) Is it possible for the J-map to factor through the canonical map $\varphi_k : C \to \mathcal{P}^1$? and if so, then what are the possibilities for the ramification of such a J-map ?

(3.2) Is it possible for the J-map to factor through a triple cover of $\mathcal{P}^1$? and if so, then What are the possibilities for the ramification of such a J-map ?

It turns out that by answering the first question, we will get a J-map ramified as in Cases 3, 4 and 5 of Table I and by answering the second question, we will get a J-map ramified as in Cases 1 and 2 of Table I.

Now we proceed to answer the question in (3. 1). In fact Theorem 3.4, gives the answer to this question.

Remark 3.3 : By applying Hurwitz's formula for the genus of a curve it is easy to show that : If C is a genus 2 curve, then the canonical map $\varphi_k : C \to \mathcal{P}^1$ is branched over exactly six points of $\mathcal{P}^1$ (since deg $(\varphi_k) = 2$).

Theorem 3.4 — Let C be a genus 2 curve. Suppose $\pi : X \to C$ is a minimal elliptic surface with a secti n and eactly one singular fiber of type I_6^*. If the associated J-map ($J : C \to \mathcal{P}^1$) factors through the canonical map $\varphi_k : C \to \mathcal{P}^1$ (say $J = \delta^\circ \varphi_k$), and if $| \{J^{-1}(1)\} | = (2, 2, 2)$, then by a suitable choice of coordinates we may assume that $\delta : \mathcal{P}^1 \to \mathcal{P}^1$ is given $\delta(x) = x^3$, and the points $\{1 = \omega^3, \omega, \omega^2, \infty\}$ must be branch points of φ_k.

Conversely if we can choose coordinates so that the points $\{1 = \omega^3, \omega, \omega^2, \infty\}$ are four of the branch points of φ_k, then the degree 6-map $J : C \to \mathcal{P}^1$ given by $J(x, y) = x^3$ factors through φ_k, and either its ramified as in Case 3 or as in Case 4 or as in Case 5 of Table I.

PROOF : Since $\deg(J) = 6$ and $\deg(\varphi_k) = 2$; therefore there is a degree 3-map $\delta : \mathcal{P}^1 \to \mathcal{P}^1$ such that $J(x, y) = \delta^\circ \varphi_k(x, y)$. By Hurwitz's formula (applied to $\delta : \mathcal{P}^1 \to \mathcal{P}^1$) we have $4 = \sum_{x \in R} (m_x(\delta) - 1)$, where $R = \{$ramification points of $\delta\}$. Since $| \{J^{-1}(\infty)\} | = 6$ and $| \{J^{-1}(0)\} | = (3, 3)$ or (6), it is clear that δ must be entirely ramified over 0 and ∞, and hence there is no other ramification of δ. Now choosing coordinates in $\mathcal{P}^1$ (= range of φ_k) so that $0 = \delta^{-1}(0)$ and $\infty = \delta^{-1}(\infty)$, we see that δ must be of the form $\delta(x) = \alpha x^3$ for some constant α and by scaling we may assume $\alpha = 1$, and hence $\delta(x) = \alpha x^3$. Now since $| \{J^{-1}(\infty)\} | = | \{\varphi_k^{-1} \delta^{-1}(\infty)\} | = 6$, and $| \{J^{-1}(1)\} | = | \{\varphi_k^{-1} \delta^{-1}(1)\} | = (2, 2, 2)$, its clear that the points $\{1, \omega, \omega^2, \infty\}$ of $\mathcal{P}^1$ (= range φ_k) must be branch points of φ_k.

To prove the converse define $\delta : \mathcal{P}^1 \to \mathcal{P}^1$ by $\delta(x) = x^3$, then clearly $\delta^{-1}(\infty) = \infty$, $\delta^{-1}(0) = 0$, and $\delta^{-1}(1) = \{1 = \omega^3, \omega, \omega^2\}$. Let $\varphi_k : C \to \mathcal{P}^1$ be the canonical map of deg 2 given by $\varphi_k(x,y)=x$. Define $J : C \to \mathcal{P}^1$ by $J(x,y) = \delta\varphi_k(x,y)=x^3$, now φ_k is branched over $\{1=\omega^3,\omega,\omega^2,\infty\}$ by assumption; therefore $\left| [J^{-1}(\infty)] \right| = (6)$, $\left| [J^{-1}(1)] \right| = (2,2,2)$ and $\left| [J^{-1}(0)] \right| = (3,3)$ or (6), depending on whether φ_k is branched over 0 or not. If φ_k is branched over 0, then there is a point η in $\mathcal{P}^1$ (= range φ_k) such that $\left| [\varphi_k^{-1}(\eta)] \right| = (2)$, with $\eta \notin \{1=\omega^3,\omega,\omega^2,\infty\}$. Let $u=\delta(\eta)$, then $\left| [J^{-1}(u)] \right| = (2,1,1,1,1)$, and hence $J=\delta^{\circ}\varphi_k$ is ramified as given in Case 5 of Table 1, if φ_k is not branched over 0, then φ_k is branched at two points say z_1 and z_2, such that $z_1, z_2 \notin \{1=\omega^3,\omega,\omega^2,\infty\}$, and of course $z_1 \neq z_2$. Let $\eta_1=\delta(z_1)$ and $\eta_2=\delta(z_2)$, hence either $\eta_1=\eta_2$ or $\eta_1 \neq \eta_2$ if $\eta_1 \neq \eta_2$, then the ramification of J must be as given in Case 3 of Table 1, and if $\eta_1=\eta_2$ let $w=\eta_1=\eta_2$, then J must be ramified as given in Case 4 of Table 1, which answers our question (3.1). Q.E.D.

Now we proceed to answer our second question (3.2).

Lemma 3.5 — Let C be a genus 2 curve, and let $\pi : X \to C$ be a minimal elliptic surface with a section and exactly one singular fiber of type I_6^*, if the associated J-map $J : C \to \mathcal{P}^1$ factors through a triple cover $\varphi : C \to \mathcal{P}^1$, then there is a degree 2 map $\psi : \mathcal{P}^1 \to \mathcal{P}^1$, such that $J=\psi^{\circ}\varphi$. Moreover we may assume that ψ is given by $\psi(x)=x^2+1$.

PROOF : Since $J=\psi^{\circ}\varphi$, deg $(\varphi)=3$ and deg$(J)=6$; therefore ψ exists and deg$(\psi)=2$. By Hurwitzs formula (applied to $\psi : \mathcal{P}^1 \to \mathcal{P}^1$), we see that ψ must be branched exactly over two points of $\mathcal{P}^1$ (= range of ψ). We know that $\left| [J^{-1}(\infty)] \right| = \left| [\varphi^{-1}\psi^{-1}(\infty)] \right| = (6)$, hence ψ is branched over ∞, also we know that if $x \in J^{-1}(1)$, then $m_x(J)$ is divisible by 2, hence ψ must be branched over 1; thus $\psi(x)=ax^2+1$, where a is a nonzero constant, so by scaling we may assume $a=1$; therefore $\psi(x)=x^2+1$. Q.E.D.

Lemma 3.6 — Let J, φ and ψ be as in Lemma 3.5, assume that $J : C \to \mathcal{P}^1$ is either ramified as in Case 1 of Table 1 or as in Case 2 of Table 1, then φ is totally ramified over exactly four points of $\mathcal{P}^1$ (= range φ). Moreover $-i$, i and ∞ must be three of these points.

PROOF : Notice that $\psi^{-1}(0)=\{-i,i\}$, $\psi^{-1}(1)=(0)$ and $\psi^{-1}(\infty)=\infty$; thus we have :

(3.6.1) $(6) = \left| [J^{-1}(\infty)] \right| = \left| [\varphi^{-1}\psi^{-1}(\infty)] \right| = \left| [\varphi^{-1}(\infty)] \right|$, hence φ is totally ramified over ∞.

(3.6.2) $(3,3)$ $\left| [J^{-1}(0)] \right| = \left| [\varphi^{-1}\psi^{-1}(0)] \right| = \left| [\varphi^{-1}(\{-i,i\})] \right|$, hence φ is totally ramified over $-i$ and i.

Now we have two cases to consider according to whether $\left| [J^{-1}(1)] \right| = (6)$ or $(2,2,2)$, if $\left| [J^{-1}(1)] \right| = (6)$, then clearly φ must be totally ramified over $\psi^{-1}(1)=0$, and hence the four branch points of φ are $\{0,-i,i,\infty\}$, if

$|[J^{-1}(1)]| = (2, 2, 2)$, then there is a $Z \in \mathcal{P}^1$ (= range φ), with $Z \notin \{0, -i, i, \infty\}$ such that $|[\varphi^{-1}(Z)]| = (3)$, let $V = \psi(Z)$, then $|[J^{-1}(V)]| = (3, 1, 1, 1)$, and hence the four branch points of φ are $\{-i, i, v, \infty\}$; thus in any case Hurwitz's formula implies that this is the only ramification of the degree 3 map $\varphi : C \to \mathcal{P}^1$. Q.E.D.

Remark 3.7 : Given a genus 2 curve C, if $\varphi : C \to \mathcal{P}^1$ is a degree three map which is totally ramified over $\{0, -i, i, \infty\}$, and if $\psi : \mathcal{P}^1 \to \mathcal{P}^1$ is given by $\psi(x) = x^2 + 1$, then $J = \psi\ 0\ \varphi$ is ramified as given in Case 1 of Table I, also if φ is totally ramified over $\{V, -i, i, \infty, V \neq 0\}$, then $J = \psi \circ \varphi$ is ramified as given in Case 2 of Table I.

So our problem now is to find a pair (φ, C) such that $\varphi : C \to \mathcal{P}^1$ is a triple cover of $\mathcal{P}^1$, which is totally ramified over exactly 4 points of $\mathcal{P}^1$, normalized to be either $\{0, -i, i, \infty\}$ or $\{-i, i, \infty, V \neq 0\}$, notice that these are the genus 2 curves C, which admit a Z_3 action (see Person[8], p. 21) given by the lifting of the action $x \to \omega x$ (where $\omega^3 = 1, \omega \neq 1$) on $\mathcal{P}^1$, hence the quotient map $\varphi : C \to C/Z_3 \cong \mathcal{P}^1$ is the desired triple cover. Moreover these curves may be written in the form

$$y^2 = (x^3 - \alpha)(x^3 - \beta)$$

and they form a 1-dimensional family of curves in the moduli space $\mathcal{M}_2$. In what follows we treat the case in which the four point are $\{0, -i, i, \infty\}$, leaving out the other case which can be treated similarly. In fact using the theory of triple covers (see Miranda[5]), I have found that there are (up to isomorphism) three such pairs, where we say that two pairs (C_1, φ_1) and (C_2, φ_2) are isomorphic if there is an isomorphism $f : C_1 \to C_2$ such that $\varphi_2 f = \varphi_1$.

Next I will present these examples (leaving out the calculations necessary to obtain them), the verification that these are indeed examples of the desired maps is straightforward, so I will leave it to the reader.

Example 3.8 — Let C_1 be the genus 2 curve given by $y^2 = x^6 - 1/4$, define $\varphi_1 : C \to \mathcal{P}^1$ by $\varphi_1(x, y) = (1-2iy) / 2x^3$, then its easy to check that $\varphi_1 : C \to \mathcal{P}^1$ is a triple cover $\mathcal{P}^1$, which is totally ramified over 0, $-i$, i and ∞.

Example 3.9 — Let C_2 be the genus 2 curve given by $y^2 = (x^\sigma + 6ix^9 - 1)/4$, define $\varphi_2 : C_2 \to \mathcal{P}^1$ by $\varphi_2(x, y) = y - (x^3 + i)$, then its easy to check that $\varphi_2 : C_2 \to \mathcal{P}^1$ is a triple cover of $\mathcal{P}^1$, which is totally ramified over 0, $-i$, i and ∞.

Example 3.10 — Let C_3 be the genus 2 curve given by $y^2 = (x^6 - 6ix^3 - 1)/4$, define $\varphi_3 : C_3 \to \mathcal{P}^1$ by $\varphi_3(x, y) = y + (x^3 + i)/2$, then its easy to check $\varphi_3 : C_3 \to \mathcal{P}^1$ is a triple cover of $\mathcal{P}^1$, which is totally ramified over 0, $-i$, i and ∞.

Remark 3.11 : Let (C_1, φ_1), (C_2, φ_2), and (C_3, φ_3) be as given in Examples 3.8, 3.9 and 3.10 respectively, define $\psi : \mathcal{P}^1 \to \mathcal{P}^1$ as in Lemma 3.5, i.e., $\psi(x) = x^2 + 1$, for $i = 1, 2, 3$, define $J_i : C_i \to \mathcal{P}^1$ by $J_i(x, y) = \psi(\varphi_i(x, y))$, then J_i is a degree 6 map ramified as given in Case 1 of Table 1. Moreover its clear that (J_1, C_1) and (J_2, C_2) are not isomorphic, but (J_2, C_2) is isomorphic to (J_3, C_3), in fact if we define $f : C_2 \to C_9$ by $f(x,y) = (-x, y)$, then f is an isomorphism; thus we have two examples of J-maps ramified as given in Case 1 of Table 1, namely J_1 and J_2.

4. The Surfaces

In this section we will describe how to get our minimal elliptic surfaces $\pi : X \to C$, assuming that each surface has asection and exactly one singular fiber of type I_6^*.

The idea here is to pull-back the rational elliptic surface defined in (1.4) via the J-maps obtained in the previous section, then use the process of "transfer of *" and the process of "deflating two *" fibers, according to the schedule in (1.5), for more details the reader should refer to Section 3 of Miranda[6].

Theorem 4.1 — Let C be a genus 2 curve, let $J : C \to \mathcal{P}^1$ be defined as in Theorem 3.4, ramified either as given in Case 3 or as in Case 4 of Table 1, and let $P : S \to \mathcal{P}^1$ be the rational elliptic surface defined in (1.4), then the pull-back of $P : S \to \mathcal{P}^1$ is a minimal elliptic surface with a section and exactly one singular fiber of type I_6^*.

Proof : Let $\pi : \; = S \times_C C \to C$ be the pull-back of $P : S \to \mathcal{P}^1$, then $\pi : X \to C$ is a minimal elliptic surface with asection and the following singular fibers (see Miranda and Persson[7], Table 7.1, p. 555) :

1. An I_6-fiber over ∞ since $N = 6$,
2. Two I_6^*-fibers over 0 since $N \equiv 3 \pmod 6$,
3. Three I_6^*-fibers over 1 since $N \equiv 2 \pmod 4$.

So in total we have 5 I_0^*-fibers; thus we use the process of "transfer of *" to transfer one * to the I_6-fiber so it becomes an I_6^*-fiber, and hence we are left with 4 I_0^*-fibers, now using the process of "deflating two *" fibers twice we get a minimal elliptic surface $\pi : X \to C$, with a section and eactly one singular fiber of type I_6^*-fiber, whose associated J-map is the degree 6 map $J : C \to \mathcal{P}^1$ defined in Theorem 3.4. Q.E.D.

Remark 4.2 : Let $J : C \to \mathcal{P}^1$ be the J-map used in Theorem 4.1, and recall that such a J-map has two different ramifications, i.e., in terms of ramification of the J-map, we have two different J- maps, and pulling-back the rational elliptic surface defined in (1.4) via each one of these J-maps as we did in Theorem 4.1 above, we get two minimal elliptic surfaces $\pi : X \to C$, each with a section and exactly one singular fiber of type I_6^*.

Remark 4.3 : If $J : C \to \mathcal{P}^1$ is ramified as in Case 5 of Table 2.2, then the pull-back $\pi : X = S \times_{\mathcal{C}} C \to C$ of the rational elliptic surface $P : S \to \mathcal{P}^1$ (defined in 1.4) is a minimal elliptic surface with a section and the following singular fibers (See Table 7.1 of Miranda and Persson[7]).

1. An I_6-fiber over ∞ since $N = 6$,

2. Three I_0^*- fibers oer 1 since $N \equiv 2 \pmod 4$.

Thus using the process of "transfer of *" and the process of "deflating two *" fibers, we transfer on * to the I_6- fiber so it becomes an I_6^*-fiber and deflate the other two *'s, hence the resulting surface is a minimal elliptic surface with asection and eactly one singular fiber of type I_6^*.

Theorem 4.4 — For $i = 1, 2$, let $J_i : C_i \to \mathcal{P}^1$ be as defined in Remark 3.11, where C_1 and C_2 are as given in Examples 3.8 and 3.9 respectively. Let $\pi_i : X_i = \underset{C_i}{S \times C_i} \to C_i$ be the pull-back of the rational elliptic surface

$P : S \to \mathcal{P}^1$ defined in (1.4), then $\pi_i : X_i \to C_i$ is a minimal elliptic surface, with a section and exactly one singular fiber of type I_6^*.

PROOF : Recall that $J_i (i = 1, 2)$ is ramified as in Case 1 of Table I. For $i = 1, 2$, let $\pi_i : X_i = \underset{C_i}{S \times C_i} \to C_i$ be the pull-back of $P : S \to \mathcal{P}^1$ via J_i, then

$\pi_i : X_i \to C_i$ is a minimal elliptic surface with asection and the following types of singular fibers (see Miranda and Perrson[7], Table 7.1).

1. An I_6-fiber over ∞ since $N = 6$,

2. Two I_0^*-fibers over 0 since $N \equiv 3 \pmod 6$,

3. An I_0^*-fibers over 1 since $n \equiv 2 \pmod 4$.

Thus we have a total of three I_0^* fibers, now using the process of "a transfer of *" and the process of "deflating two *" fibers we get a minimal elliptic surface $\pi_i : X_i \to C_i$ $(i = 1, 2)$, with asection and exactly one singular fiber of type I_6^*.

Remark 4.5 : If the J-map is ramified as in Case 2 of Table I, then the pull-back of the rational elliptic surface ($P : S \to \mathcal{P}^1$) defined in (1.4) is a minimal elliptic surface, with a section and the following singular fibers (see Miranda and Persson[7], Table 7.1)

1. An I_6-fiber over ∞ since $N = 6$,

2. Two I_0^*-fibers over 0 since $N \equiv 3 \pmod 6$,

3. Three I_0^*-fibers over 1 since $N \equiv 2 \pmod 4$.

Hence using the process of "transfer of *" and the process of "deflating two *" fibers twice, we get a minimal elliptic surface with a section and exactly one singular fiber of type I_6^*.

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