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EFFECT OF LINEARITY ON THE STRUCTURE OF SHOCK WAVES IN DUSTY GASES

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The structure of fully dispersed waves in dusty gases is investigated using a physical model with one slow and one fast relaxation mode. This model is obtained formally by neglecting the heat conductivity ($\lambda = 0$). Exact solution is obtained formally by linearizing the system of autonomous nonlinear differential equations.

1. INTRODUCTION

Flowing gases with solid particles are called dusty-gas flows. In dusty gas there is generally a relative motion between phases and in many cases there is also heat and mass transfer between the solid particles and surrounding gas. Solid particles in gases are generally denoted as the dispersed phase, and particles of different sizes should be treated as different phases. The size of particles in practical dusty-gas flows can vary markedly from a few tenths of a micron (tobacco smoke) to the order of millimeters (coal and sand particles).

Although particles can have different geometrical shapes, they are generally assumed or approximated as spherical, which is usually sufficient to obtain a basic qualitative understanding of the general behaviour of nonspherical or even deformed particles. In general, solid particles are inert, rigid, and preserve their shape even in the case of collisions between particles and when extremely large normal and shear stresses exist in the gaseous phase. In actual dusty-gas flows the collisions between particles may result in agglomeration which would increase the effective size of particles.

A stationary or moving shock wave in dusty gas disrupts the equillibrium of temperature and velocity between the phases, because both energy (heat) and momentum transfer between the gas and particles does not occur instantaneously[1-3].

Over the past thirty years a lot of research has been conducted to better understand and predict dusty-gas flows with shock waves[1-8], it has been assumed in these research that the shock structure in dusty-gases can be separated into two regions, namely gas-dynamic shock wave and relaxation zone. The transport coefficients of viscosity and heat conductivity have been neglected in all previous papers, so that the shock structure appeares as a discontinuity.

Fully dispersed shock waves in steady and unsteady dusty-gas flows have smooth continuous changes in gas flow properties instead of a discontinuous front, like those for the particles.

Studies of such waves have also been done[1, 2, 4-10]. A difficulty exists in the calculation of steady dispersed shock waves, with the selection of unique initial conditions for starting the numerical integration[1,2].

The main purpose of this paper is to present an exact solution for a shock wave in dusty gas with both viscosity and relaxation time to find the Effect of this linear solution on the shock wave Profile.

The equilibrium properties of a dusty gas can be reduced formally to the equilibrium properties of a simple gas by introducing certain parameters of the mixture, for example the Mach number of the mixture[7] :

$$\overline{M} = M \frac{1-\varphi}{\sqrt{1-\mu}} \sqrt{\frac{1+(\mu/(1-\mu))\,(c/c_v)}{1+(\mu/(1-\mu))\,(c/c_p)}} \qquad \ldots (1)$$

where μ is the mass-fraction of the particles, c the specific heat of the particle material, c_v and c_p the specific heats of the gas at constant volume and pressure respectively. The volume fraction of the particles φ will be neglected in the present paper. The Mach number of the gas M is based on the gas velocity and on the speed of sound far upstream.

For a simple gas a shock wave can occur only if one has for the Mach number of the gas $M > 1$. The Mach number of the mixture $\overline{M}$ becomes unity for :

$$M = M_{min} = \sqrt{1-\mu} \sqrt{\frac{1+(\mu/(1-\mu))\,(c/c_p)}{1+(\mu/(1-\mu))\,(c/c_v)}} \,. \qquad \ldots (2)$$

For Mach numbers in the range from $M = M_{min}$ to $M = 1$ the Rankine-Hugoniot conditions predict a new equilibrium state which can be realized by fully dispersed waves. The changes of the thermodynamic state of the gas are caused by four relaxation processes : by the molecular processes of momentum and energy transfer (viscosity and heat conductivity) and by the macroscopic processes of friction and heat transfer between gas and suspended particles. The characteristic time for the molecular processes in the mean time between molecular collisions :

$$\tau = l/\bar{c} \qquad \ldots (3)$$

where l is the mean free path of the gas and $\bar{c} = (8K\,T/\pi\, m)^{1/2}$ is the mean molecular velocity. The characteristic time for the relaxation of the macroscopic velocity[5] is

$$\tau_v = \frac{m_P}{3\pi\sigma_P\eta} = \frac{16}{45\pi}\,\frac{\sigma_P^2}{l^2}\,\frac{\rho_P}{\rho_D}\,\frac{l}{\bar{c}} \qquad \ldots (4)$$

and the characteristic time for the relaxation of the temperature of the gas is

$$\tau_T = \frac{m_P C_P}{2\pi \sigma_P \lambda} = \frac{16K}{75\pi} \frac{\sigma_P^2}{l^2} \frac{\rho_P}{\rho_D} \cdot \frac{l}{\bar{c}} \qquad \dots (5)$$

where m_P is the mass of a particle, η the viscosity, λ the heat conductivity, σ_P the diameter of the spherical particles, ρ_P the mass density of the particle material, ρ_D the mass density of the gas, K the ratio of the specific heats. It has been assumed that the viscosity and the heat conductivity of the gas may be described by the gas kinetic model of rigid spheres.

Equations (4) and (5) show that $\tau_V \approx \tau_T$. In order to obtain an analytical solution for the structure of dispersed waves a model is used which takes two relaxation processes into account : one molecular relaxation process described with the viscosity of the gas η and one macroscopic relaxation process described with the relaxation time τ_V. This model is equivalent to assuming for the heat conductivity of the gas $\lambda = 0$. This assumption has been made by Taylor[15] in his calculation of the shock wave structure in simple gases.

2. Basic Equations

For the one-dimensional steady flow of a dusty gas the continuity equation, the momentum equation and the energy equation are

$$\frac{d(u_D \zeta_D)}{dx} = 0 \qquad \dots (7)$$

$$\frac{d(u_P \zeta_P)}{dx} = 0 \qquad \dots (8)$$

$$\zeta_D u_D \frac{du_D}{dx} = F_{Px} - \frac{dp_{xx}}{dx} \qquad \dots (9)$$

$$\zeta_P u_P \frac{du_P}{dx} = -F_{Px} \qquad \dots (10)$$

$$\zeta_D u_D \frac{de_D}{dx} = -p_{xx} \frac{du_D}{dx} + Q_P + F_{Px}(u_P - u_D) - \frac{dq_x}{dx} \qquad \dots (11)$$

$$\zeta_P u_P \frac{de_P}{dx} = -Q_P. \qquad \dots (12)$$

The subscripts P and D refer to the particles and to the gas respectively. The quantity u represents the velocity, ζ the density (i.e. the mass of the gas or the particles per unit volume of the system), F_{Px} the force per unit volume exerted by the particles on the gas, e the internal energy, Q_P the heat transferred from the particles to the gas per unit time and per unit volume, q_x the heat flux in the gas and p_{xx} one component of the stress tensor. Equations (7)-(12) have to be solved for the boundary conditions :

$$u_P = u_D = u_0, \quad T_P = T_D = T_0 \text{ for } x \to -\infty$$

and

$$u_P = u_D = u_1, \quad T_P = T_D = T_1 \text{ for } x \to +\infty.$$

The interaction force F_{P_x} and heat Q_P exchanged between particles and gas are eliminated by adding the momentum equations to (11) and (12). The resulting two equations can be integrated once immediately. By introducing the integrated continuity equations :

$$\zeta_D u_D = m_1 = m \qquad \ldots (13)$$

and

$$\zeta_P u_P = m_2 = \beta m \qquad \ldots (14)$$

one obtains :

$$\zeta_D u_D^2 + \zeta_P u_P^2 + p_{xx} = P \qquad \ldots (15)$$

$$(e_D + u_D^2/2) + \beta (e_P + u_P^2/2) + p_{xx} u_D/m + q_x/m = E \qquad \ldots (16)$$

where the abbreviation $\beta = \mu/(1-\mu)$ has been used. For further treatment explicit expressions for the equations of state, the stress tensor and the heat flux have to be introduced :

$$e_D = c_v T_D, \quad e_P = c T_P, \qquad \ldots (17)$$

$$p_{xx} = p - \frac{4}{3}\eta \frac{du_D}{dx}, \quad q_x = -\lambda \frac{dT_D}{dx} \qquad \ldots (18)$$

$$p = \rho_M R_M T \quad \text{or} \quad p = \zeta_D R T_D \ (\phi << 1). \qquad \ldots (19)$$

Using these expressions eqns. (15) and (16) can be put in the following form :

$$\frac{4\eta}{3m}\frac{du_D}{dx} = u_D + \beta u_P + \frac{R T_D}{u_D} - \frac{P}{m}, \qquad \ldots (20)$$

$$\frac{\lambda}{m}\frac{dT_D}{dx} = (c_v T_D + u_D^2/2) + \beta (c T_P + u_P^2/2) + u_D (P/m - u_D - \beta u_P) - E. \qquad \ldots (21)$$

The integration constants P and E can be expressed in terms of the variables of state ahead of the shock wave :

For the special case of simple gas, i.e. for $\beta = 0$, these equations are of course identical with Gilbarg and Paolucci's[17] equations for the shock wave in a simple gas. The system of four differential equations (26)-(29) is difficult to solve numerically because of the nonlinearities and the two singularities of the direction field in phase space. These singularities correspond to the equilibrium conditions ahead of and behind the shock wave. In our previous paper, this system is solved analytically in phase space by expanding the variables of state in power series[11].

In order to simplify this solution a model with one molecular and one macroscopic relaxation parameter is used. This model is obtained by setting $\lambda = 0$. It follows $(1/\tau_T) = 0$ and from equation (29) one finds θ_P = const. θ_0. Using (27) which becomes purely algebraical the temperature θ_D can be eliminated from equation (26) :

$$5\ \bar{\eta}\ \frac{dw_D}{dx} = 6\, w_D^2 + 7\,\beta\, w_P\, w_D - \beta\, w_P^2 - 7\, w_D + 5F. \qquad \text{... (35)}$$

$$\bar{\tau}_v\ w_P\ \frac{dw_P}{dx} = w_D - w_P. \qquad \text{... (36)}$$

Here the following abbreviations have been used :

$$K = c_P/c_v = 7/5\,,\quad \delta = (K-1)/2 \qquad \text{... (37)}$$

$$A' = \theta_0\,\beta\, c/c_v\,,\qquad F = \delta\,(1+\alpha) - A. \qquad \text{... (38)}$$

3. EQUILIBRIUM

Far in front of the wave and far behind the wave all gradients of the variables of state γ become zero. Under this condition the equilibrium state can be calculated from eqns. (35) and (36) :

$$w_{0,1} = \frac{7 \pm \varepsilon}{12(1+\beta)}\,, \qquad \text{... (39)}$$

$$\theta_{0,1} = \frac{35 - \varepsilon^2 \mp 2\,\varepsilon}{144(1+\beta)}\,, \qquad \text{... (40)}$$

$$\phi_{0,1} = \frac{5 \mp \varepsilon}{12}\,, \qquad \text{... (41)}$$

where

$$\varepsilon^2 = 49 - 120\,(1+\beta)F. \qquad \text{... (42)}$$

The parameter ε is a measure for the strength of the change in the variables of state. One has

$$M_0^2 = \frac{1}{1+\beta}\ \frac{1+\varepsilon/7}{1-\varepsilon/5}\,. \qquad \text{... (43)}$$

For very strong shock waves with $M_0 \to \infty$ one has $\varepsilon \to 5$.

The limit $\varepsilon \to 0$ for very weak shock waves is obtained for the Mach number $M_0 = (1+\beta)^{-1/2}$. Fully dispersed wave occur for $(1+\beta)^{-1/2} < M_0 < 1$. In equation (38) it has been assumed that $c = c_p$.

4. Phase Space

Eliminating the spatial coordinate x from differential equations (35) and (36) one obtains the single differential equation :

$$\frac{5\,\bar{\eta}}{\bar{\tau}_v} \frac{dw_D}{dw_P} = \frac{M(w_P, w_D)}{L(w_P, w_D)} \qquad \dots (44)$$

where

$$M(w_P, w_D) = w_P\,(6\,w_D^2 + 7\,\beta\,w_P\,w_D - \beta\,w_P^2 - 7w_D + 5F) \qquad \dots (45)$$

$$L(w_P, w_D) = w_D\,(w_D - w_P). \qquad \dots (46)$$

The direction field given by eqn. (44) in the w_D, w_P-plane has three singular points :

$$P_{00} = (w_D = 0,\ w_P = 0)\ ,\quad P_0 = (w_D = w_P = w_0)\quad \text{and}\quad P_1 = (w_D = w_P = w_1).$$

The points P_0 and P_1 which correspond to the equilibrium conditions in front of the shock wave and behind the shock wave are given as intersections of the hyperbola $M/w_P = 0$ and the straight line $L/w_D = 0$. It can be shown that P_0 is always a saddle point and P_1 a node. The changes of the variables of state in the dispersed wave are represented by the integral curve connecting the singularities which correspond to the equilibrium states in front of the wave and behind the wave. The following transformation of the w_D, w_P-plane is introduced :

$$w_P = w_0 - \psi\ ,\quad w_D = w_0 - \nu \qquad \dots (47)$$

where ψ, ν are very small positive quantities. Introducing (47) into eqns. (35), (36) and (44) and neglecting the small quantity of the second order, one obtains

$$\bar{\eta}\,\frac{d\nu}{dx} = A\,\nu + \beta\,\psi \qquad \dots (48)$$

$$\bar{\tau}_v\,w_0\,\frac{d\psi}{dx} = \nu - \psi \qquad \dots (49)$$

$$\frac{\bar{\eta}}{\bar{\tau}_v\,w_0}\,\frac{d\nu}{d\psi} = \frac{A\nu + \beta\psi}{\nu - \psi} \qquad \dots (50)$$

where

$$A = \frac{12\,w_0 + 7\,\beta\,w_0 - 7}{5\,w_0}.$$

By integrating eqn. (50), one obtains the result in the form :

$$\ln \frac{\nu}{h} = \frac{g(A+\beta)}{gA}(1-z) + \left[1 + \frac{g(A+\beta)}{(gA)^2} \right]$$

$$[\ln(1-gA) - \ln(1-gA\,z)] \qquad \ldots (51)$$

where $h = w_0 - w_1$, $g = \bar{\tau}_v\, w_0/\bar{\eta}$, $z = \frac{\psi}{\nu}$,

which is the phase space for the shock wave in ν, ψ plane.

5. The Shock Profile

Inserting the previous result of eqn. (51) into the differential equation (48), it follows that :

$$\bar{\eta}\,\frac{d\nu}{dx} = -\frac{\beta}{hgA}\,[\,\nu\,(h-\nu)\,]. \qquad \ldots (52)$$

By integrating, one obtains

$$\nu = \frac{h}{2}\left[1 + \tanh\left(-\frac{\beta x}{2bA} \right)\right] \qquad \ldots (53)$$

where $b = \bar{\tau}_v\, w_0$, $A < 0$.

By using eqn. (47), one finds

$$w_D = w_0 - \frac{h}{2}\,[1 + \tanh(Bx)\,] \qquad \ldots (54)$$

where $B = -\frac{\beta}{2bA}$ is a positive quantity.

from eqns. (49) and (53), one obtains

$$b\,\frac{d\psi}{dx} = \frac{h}{2}\,[1 + \tanh(Bx)] - \psi. \qquad \ldots (55)$$

By integrating, one finds

$$\psi = \frac{h}{2}\left[1 + \tanh(Bx) + \frac{\beta}{2A} \right]. \qquad \ldots (56)$$

By using eqn. (47), one obtains :

$$w_P = w_0 - \frac{h}{2}\left[1 + \tanh(Bx) + \frac{\beta}{2A} \right]. \qquad \ldots (57)$$

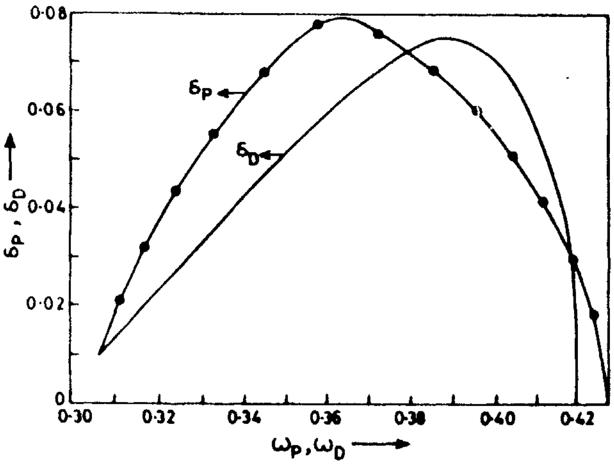

FIG. 1 Local deviation between exact (numerical) solution and analytical solution for $M_0 = 0.97$ and $B = 0.6$. $\delta_{P_{ana}} = (\omega_{P_{ana}} - \omega_{P_{num}})/\omega_{P_{num}}$ and $\delta_D = (\omega_{D_{ana}} - \omega_{D_{num}})/\omega_{D_{num}}$

Equations (54) and (57) represent linear solutions for the shock wave profile.

The numerical solutions of eqns. (35), (36) has been obtained to compare the previous result with the numerical result and one find that the maximum error is less than 8%.

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