Indian J. pure appl. Math., **24**(2) : 87-90, February 1993

CONGRUENCE PROPERTIES OF A CLASS OF ARITHMETIC FUNCTIONS

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(*Received* 9 *June* 1992; *after revision* 15 *October* 1992; *accepted* 29 *October* 1992)

Harris and Subbarao[2] established certain congruence properties of $\sigma_r(n)$, the sum of the rth powers of the positive divisors of n, which give the results of ramanathan[3] and Gupta[1] as a particular case. In this paper we have made a study of some congruence properties of arithmetic functions from an infinite class $\mathcal{f}$ which includes σ_r as a member.

1. Introduction

Suppose $\sigma(n)$ denotes the sum of the positive divisors of n. Using elliptic theta functions and allied infinite series Ramanathan[3] has proved that the congruence $\sigma(kn-1) = O \pmod{k}$ holds for all $n > 0$, when $k = 3, 4, 6, 8, 12$ or 24. Gupta[1] has shown that these are the only congruences of this type. Subbarao[4] gave very elementary proofs of these congruence properties. Later Harris and Subbarao[2] have established certain congruence properties of $\sigma_r(n)$, the sum of the rth powers of the positive divisors of n, which include the earlier stated results on $\sigma(n)$ as special cases.

In this paper we made a study of some congruence properties of arithmetic functions in an infinite class $\mathcal{f}$, which includes σ_r as a member.

We describe the class $\mathcal{f}$ of arithmetic functions in section 2 and in section 3 establish the congruence properties for functions f in the class $\mathcal{f}$.

2. The Class $\mathcal{f}$ of Functions

For any integer $k > 0$, let $\mathcal{f}_k$ be the class of all integer-valued arithmetic functions satisfying the following conditions:

$$f(mn) = f(m) f(n) \text{ for all } m \text{ and } n; \qquad \ldots (2.1)$$

$$f(n + k) = f(n) \pmod{k} \text{ for all } n. \qquad \ldots (2.2)$$

It is well known that a function f satisfying (2.1) is a completely multiplicative function and that such a function is determined by its values at each prime number.

Note that $f(n) = n^r$, where $r \geq 0$ is an integer, is in the class $\mathscr{C}_k$ for any $k > 0$. It follows from a result of Subbarao[5] that any non-identically zero function in $\bigcap_{k>0} \mathscr{C}_k$ is of the form $f(n) = n^r$.

However we observe that each $\mathscr{C}_k$ has infinitely many functions. In fact, $\mathscr{C}_1$, being the class of all completely multiplicative functions, is infinite. If $k > 1$, the infinitude of the class $\mathscr{C}_k$ can be established in several ways. For example, for a given $k > 1$, let $p_1, p_2, \ldots, p_r$ be the primes upto k. Let f be a completely multiplicative function for which $f(p_i) = b_i$ where $b_i \equiv 1 \pmod{k}$ for $i = 1, 2, \ldots, r$ and then for any prime $q > k$, we have $v = q - k[q/k] < k$ so that the prime factors of v are among $p_1, p_2, \ldots, p_r$, say $v = \prod_{i=1}^{r} p_i^{\alpha_i}$, for some integers $\alpha_i \geq 0$, and now define $f(q) = dq$, where $dq \equiv \prod_{i=1}^{r} b_i^{\alpha_i} \pmod{k}$, we find $f \in \mathscr{C}_k$. Since b_i $(i = 1, 2, \ldots, r)$ are chosen arbitrarily, subject to $b_i \equiv 1 \pmod{k}$, we construct infinitely many functions f in the class $\mathscr{C}_k$.

We thank the referee for suggesting the following simple way of constructing infinitely many functions in the class $\mathscr{C}_k$: Let $f(p) = c_p$ (c_p are positive integers $\equiv 1 \pmod{k}$) and $f(p^r) = c_p^r$ for all primes p. Define $f(n) = \pi c_p^{\alpha}$, where $n = \pi p^{\alpha}$. Then $f \in \mathscr{C}_k$.

Let $\mathscr{C} = \bigcup_{k>0} \mathscr{C}_k$. For each $f \in \mathscr{C}$, we define its 'associate function', F_f by $F_f(n) = \sum_{d|n} f(d)$ where the sum is over all positive divisors of n.

For example, the associate function of $f(n) = n^r$ is $\sigma_r(n)$.

Let $\mathscr{f}$ be the class of all associate functions of $f \in \mathscr{C}$. That is, $F \in \mathscr{f}$ if and only if $F(n) = F_f(n)$ for some $f \in \mathscr{C}$.

3. Main Results

In this section we establish certain congruence properties of functions F in the class $\mathscr{f}$. In all that follows k is a fixed integer ≥ 3, $f \in \mathscr{C}_k$ is such that $(f(n), k) = 1$ for all $n \geq 1$ and $F(n) = F_f(n)$ is its associate function.

We introduce some terminology and notation. If d is a positive divisor of n and $dd' = n$ then $f(d) + f(d')$ or $f(d)$ is called a component of $F(n)$ according as $d \neq d'$ or $d = d'$. If every component of $F(n)$ is congruent to an integer a modulo k, we say that $F(n)$ is componently congruent to a mod k and write in this case $F(n) \equiv a \pmod{k}$. Observe that neither the component congruence need imply the congruence nor the congruence need imply the component congruence.

Theorem 3.1 —Suppose k, L are fixed positive integers with $k \geqslant 3$ and $(L, k) = 1$. For any integer $a \geqslant 0$ we have

$$F(nk + L) \equiv a \pmod{k} \text{ for all } n \geqslant 0 \qquad \ldots (3.1)$$

if and only if all the following conditions hold

$$L \text{ is a quadratic non-residue of } k \qquad \ldots (3.2)$$

$$1 + f(L) \equiv a \pmod{k} \qquad \ldots (3.3)$$

$$[f(w) - 1][f(w) + 1 - a] \equiv O \pmod{k} \text{ for all } w \text{ with } (w, k) = 1. \qquad \ldots (3.4)$$

PROOF: First suppose $F(nk + L) \equiv a \pmod{k}$.

If possible, L is a quadratic residue of k. Then we can find an integer q such that $q^2 \equiv L \pmod{k}$ so that both q^2 and $(k+1)\, q^2$ are in the sequence $nk + L$. Hence $F(q^2) \equiv a \pmod{k}$ and $F[(k+1)q^2] \equiv a) \bmod k$. Since $f(q)$ is a component of $F(q^2)$ and $f(q) + f((k+1)q)$ is a component of $F[(k+1)q^2]$ we get $f(q) \equiv a \pmod{k}$ and $f(q) + f(k+1)\, f(q) \equiv a \pmod{k}$, wherein we have used (2.1). Therefore $f(k+1)\, f(q) \equiv O \pmod{k}$, which is contradiction, since $(f(n), k) = 1$ for all $n \geqslant 1$. Thus (3.2) holds.

$n = 0$ in (3.1) gives $F(L) \equiv a \pmod{k}$. Also, by (3.2), $L \neq 1$ so that $f(1) + f(L)$ is a component of $F(L)$. Therefore (3.3) holds.

If w is an integer with $(w, k) = 1$, there exists an integer $w_1 \neq w$ such that $w w_1 \equiv L \pmod{k}$. Then by (3.1), $F(w w_1) \equiv a \pmod{k}$ and therefore we have

$$1 + f(w w_1) \equiv a \pmod{k} \text{ and } f(w) + f(w_1) \equiv a \pmod{k}.$$

Using (2.1) in the first of these congruences and eliminating $f(w_1)$ from them we get (3.4).

Conversely suppose (3.2), (3.3) and (3.4) hold. Let d be a positive divisor of $N = nk + L$ and $d d' = N$. In view of (3.2), $d = d'$ cannot hold so that any component of $F(N)$ is of the form $f(d) + f(d')$ only. Now since $(d, k) = 1$, we have by (2.2) and (3.4), that

$$f(d)[f(d) + f(d')] = f(d)^2 + f(N) \equiv a f(d) + 1 - a + f(L) \pmod{k}$$

This, by a use of (3.3), gives

$$f(d)[f(d) + f(d')] \equiv a f(d) \pmod{k}$$

from which it follows $f(d) + f(d') \equiv a \pmod{k}$, since $(f(d), k) = 1$. Thus $F(nk + L) \equiv a \pmod{k}$ for all $n \geqslant 0$.

Theorem 3.2 —Suppose $(L, k) = 1$ and $F(nk + L) \equiv a \pmod{k}$ for all integers $n \geqslant 0$. Then $F(nk + L \equiv a \pmod{k})$ for all integers $n \geqslant 0$ if and only if $a \equiv O \pmod{k}$.

PROOF: Suppose both $F(nk + L) \equiv a \pmod{k}$ and $F(nk + L) \equiv a \pmod{k}$ hold. For any prime p with $(p, k) = 1$, we can find, by Dirichlet theorem on primes in an arithmetic progression, another prime q such that $pq \equiv L \pmod{k}$. Then each of the two components of $F(pq)$ is congruent to a mod k so that their sum $F(pq) \equiv 2a$

(mod k). But $F(pq) \equiv a \pmod{k}$, by hypothesis. Hence $2a \equiv a \pmod{k}$, proving $a \equiv O \pmod{k}$.

Conversely if $a \equiv O \pmod{k}$ then $F(nk+L) \equiv a \pmod{k}$ shows each component of $F(nk+L)$ is congruent to zero mod k and hence $F(nk+L) \equiv O \pmod{k} \equiv a \pmod{k}$.

Remark 3.1: In the special case of $f(n) = n^r$, Theorems 3.1 and 3.2 respectively give Theorems 1 and 3 of Harris and Subbarao[2].

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