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QUATERNIONS AND CONJUGATE SYSTEM OF TETRAHEDRA

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Given a point P not on the faces of the tetrahedron $(A) = A_0A_1A_2A_3$, the lines A_iP meet the opposite faces at the point A_i'. The two tetrahedra (A) and (A') are perspective from the point P where the plane of perspectivity p is called the polar of P for (A), P is the pole of p. Two tetrahedra (B) and (C) are called conjugate with respect to (A) if the polars of the vertices of one with respect to (A) form the faces of the other and vice versa. It is shown that given a point not on the face of (A), a conjugate pair of tetrahedra, with the given point as the vertex of one, can be obtained using quaternions. Two parallel but independent conjugate systems can be obtained into which the Mobius structures and the desmic structures are introduced using the quaternions and some interesting properties of these systems have been studied.

1. NOTATIONS AND PRELIMINARIES

A field $\mathcal{F}$ is called formally real if for $a_i \in \mathcal{F}$, $i = 1, \ldots, n$, $\sum_{i=1}^{n} a_i^2 = 0$ for all i (Macduffee[5]). So if a field is formally real then its characteristic is zero. Let $\mathcal{P}_3$ be a three dimensional projective space over a formally real field $\mathcal{F}$. Throughout the paper our field $\mathcal{F}$ is formally real. Let $(X) = X_0X_1X_2X_3$, $X = A, B$ be two tetrahedra in $\mathcal{P}_3$. The homogeneous coordinates in the vertices of (B) referred to (A) and some unit point form the rows of a 4×4 matrix M (Sanyal[9]). The homogeneous coordinates (x_i) and (ω_i), $i \in I_3$ (Index set containing the numbers 0 to 3) and $t(\neq 0) \in \mathcal{F}$, represent the same point (Artzy[1], p. 127) so for every nonsingular diagonal matrix $D = \mathrm{diag}(d_0, d_1, d_2, d_3)$, DM also represents the coordinates of vertices of (B). The matrices M and N are defined to be equivalent if $N = DM$ for some nonsingular diagonal matrix D. This equivalence relation gives equivalence classes where rows of any matrix of a particular class represent the vertices of a particular tetrahedron for another. In symbol it is written as $(B)_A = M = (\!(b_{ij})\!)$ to represent (B) for (A). So if $(B)_A = (\!(b_{ij})\!)$, $(A)_C = (\!(a_{ij})\!)$ then $(B)_C = (\!(d_{ij})\!) = (\!(b_{ij})\!)(\!(a_{ij})\!)$.

It is symbolised as $(B)_C = (B)_A(A)_C$. In this case $(B)_B$ is represented as the identity matrix. So $(A)_B = [(B)_A]^{-1} = (\!(b_{ij})\!)^{-1}$. Here any matrix N, for which $MN = NM = t\mathrm{I}$; $t(\neq 0) \in \mathcal{F}$ serves the purpose of M^{-1}.

Note – 1: The matrix of a tetrahedron can be multiplied by a nonsingular diagonal matrix from the left without changing the tetrahedron. But it cannot be done

when the matrix is used for the coordinate transformation as in (1.1). The matrix of the tetrahedron have to be fixed throughout the work when it is used for the coordinate transformation. But however one can multiply it by a nonzero scalar to make the matrix, to be used for a coordinate transformation, simpler.

Two tetrahedra are called perspective if the lines $A_i B_i$ (lines joining the vertices A_i and B_i) are concurrent at some point P. In that case the lines common to corresponding planes are on a plane p (Court[2]). P and p are called the centers of perspectivity and the plane of perspectivity respectively (Gerber[4]). It is denoted as $(A) \overset{P}{\underset{p}{=}} (B)$ [Coxeter[3], p. 21]. Here

$$(B)_A = \begin{bmatrix} a_0 & p_1 & p_2 & p_3 \\ p_0 & a_1 & p_2 & p_3 \\ p_0 & p_1 & a_2 & p_3 \\ p_0 & p_1 & p_2 & a_3 \end{bmatrix} \equiv \frac{(a_0, a_1, a_2, a_3)}{(p_0, p_1, p_2, p_3)} \qquad \ldots (1.2)$$

where the parameters a_0, a_1, a_2 and $a_3 \in \mathcal{F}$ and the co-ordinates of P for (A) are (p_0, p_1, p_2, p_3) (Sanyal and Misra[10]).

Definition 1.1—Let $(A) \overset{P}{\underset{p}{=}} (B)$. If $B_i \in a^i$, the opposite plane of $A_i \forall i$ then p is called the harmonic polar (or polar) of P and P is called the harmonic pole (or pole) of p for (A). If $P = (p_i)$ (where each $p_i \neq 0$ as P is not on any face of (A)) then the equation of p is

$$\sum_{i=0}^{3} x_i p_i^{-1} = 0 \qquad \ldots (1.3)$$

If the polar of a point $P(p_i)$ with respect to (A) passes through the point $Q(q_i)$ then

$$\sum_{i=0}^{3} \frac{q_i}{p_i} = 0 \qquad \ldots (1.4)$$

For a matrix $M = ((m_{ij}))$, $m_{ij} \neq 0 \ \forall_{i,j}$, the matrix M^* is defined to be the matrix obtained by taking the reciprocal of each entry of M, i.e., $M^* = ((m_{ij}^{-1}))$. So if (p_i) and (q_i) are taken to be the row matrices, (1.4) can be reduced to the following

$$(q_i)[(p_i)^*]^T = \theta \qquad \ldots (1.5)$$

Definition 1.2—If

$$(q_i)[(p_i)^*]^T = \theta = (p_i)[(q_i)^*]^T \qquad \ldots (1.6)$$

then P and Q are said to be the conjugate points of (A).

Let $(B)_A = ((b_{ij}))$. If

$$((b_{ij}))[((b_{ij}))^*]^T = \text{diag}(d_0, d_1, d_2, d_3),\ d_0 d_1 d_2 d_3 \neq \theta, \qquad \ldots (1.7)$$

then the vertices B_k and $B_l (k \neq l)$ are the conjugate points and (B) is called the self polar for (A). In this case $((b_{ij})\{(b_{ij}^{-1})) = 4I$. So $(A)_B = ((b_{ij}^{-1}))^1$ which implies that (A) is the self polar for (B). Thus (A) and (B) are called the pair of mutually self polar tetrahedra if (1.7) is satisfied (Konnuly[5]).

Definition 1.3—If $(A), (B)$, and (C) be three tetrahedra in $\mathscr{P}_3$ where $(B)_A = ((b_{ij}))$ and $(C)_A = ((c_{ij}))$. (C) is called the polar tetrahedra of (B) for (A) if and only if

$$((c_{ij}))\{((b_{ij}))^{\#}\}^{\mathrm{T}} = \mathrm{diag}(r_0, r_1, r_2, r_3),\ r_0 r_1 r_2 r_3 \neq 0. \qquad \ldots (1.8)$$

Definition 1.4—A pair of mutually inscribed, circumscribed or interlocked tetrahedra is said to be Mobius (Maxwell[7], p. 18).

If (A) and (B) form a Mobius pair such that X_i lies on the plane $Y_j Y_k Y_l (X, Y = A, B, i \neq j \neq k \neq l, i, j, k, l \in I_3)$ then

$$(B)_A = \begin{bmatrix} 0 & a & -b & c \\ -a & \theta & d & c \\ b & -d & \theta & a \\ -c & -e & -g & \theta \end{bmatrix} \text{(Maxwell[7], p. 18)} \qquad \ldots (1.9)$$

Definition 1.5—Two tetrahedra (A) and (B) are said to be desmic if they are fourfold perspective (Court[2], p. 273; Maxwell[7], p.21). If

$$(B)_A = \frac{(-a,\ b,\ c,\ d)}{(\ a,\ -b,\ -c,\ -d)} \qquad \ldots (1.10)$$

then (A) is perspective with each of (B), $B_1 B_0 B_3 B_2$, $B_2 B_3 B_0 B_1$ and $B_3 B_2 B_1 B_0$. The four centres of perspectivities of these four perspectivities form the vertices of a third tetrahedron (C) given as

$$(C)_A = \frac{(-a, -b, -c, -d)}{(\ a,\ b,\ c,\ d)} \qquad \ldots (1.11)$$

$(A), (B)$ and (C) form a desmic system of tetrahedra and are said to be in a desmic system.

2. Quaternions and Tetrahedra

Let $\mathscr{Q}$ be the set of all quaternions over $\mathscr{F}$ (Macdufee[6], p. 251-56). Let $\alpha \in \mathscr{Q}$ where $\alpha = a + bi + cj + dk$, $a, b, c, d \in \mathscr{F}$.

a, b, c and d are called the components of the quaternion α. The conjugate of α is denoted by $\bar{\alpha}$ is the quaternion $a - bi - cj - dk$. $\alpha\bar{\alpha} = N(\alpha) = a^2 + b^2 + c^2 + d^2$ is called the norm of α. For $N(\alpha) \neq 0$

$$\alpha^{-1} = \bar{\alpha}/N(\alpha).$$

As $\mathcal{F}$ is formally real, every non-zero quaternion in this paper has a non-zero norm and hence has an inverse.

There exist an isomorphism between the quaternions and the fourth order matrices, over the same field $\mathcal{F}$, by the following correspondence.

$$a+bi+cj+dk \leftrightarrow \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \qquad \ldots (2.1)$$

A matrix of this form is called a quaternion matrix (Mirsky[8], p. 268). If the quaternion is not zero then the corresponding matrix is nonsingular. The transpose of the above matrix corresponds to the conjugate of the corresponding quaternion.

If for a pair of tetrahedra (A) and (B), $(B)_1$ happens to be a quaternion matrix then the matrix can be replaced by the quaternion to make expression simple. For this purpose two quaternions are equivalent if one is obtained by multiplying the other by a non-zero $r \in \mathcal{F}$ because they will represent the same tetrahedron for a given reference frame.

If $(B)_1 = \alpha$, then $(A)_B = \alpha^{-1}$. But here $\bar{\alpha}$ and α^{-1} are equivalent as $\bar{\alpha} = N(\alpha)\alpha^{-1}$. So $(A)_B = \bar{\alpha}$.

Let $(B)_1 = \alpha = p+qi+rj+sk$. By multiplying a suitable diagonal matrix to the matrix of α one can get a matrix of the form (1.10) if and only if $p^2 = q^2 = r^2 = s^2$. So if $(B)_1 = p+qi+rj+sk$ then (A) and (B) are desmic if and only if

$$p^2 = q^2 = r^2 = s^2 \qquad \ldots (2.2)$$

Again if $(B)_1 = a+bi+cj+dk$ then (A) and (B) are Mobius in some order if and only if

$$\text{one of } a, b, c, d \text{ is zero.} \qquad \ldots (2.3)$$

This is because when one of a, b, c, d is zero then it is possible to change the quaternion matrix, by some row interchanges, to a skew-symmetric matrix to take the form (1.9). Conversely if quaternion matrix is skew-symmetric then the first component has to be zero.

Theorem 2.1 — Let α and β be two quaternions such that both of them has one particular component (except the first) equal to zero. If $\alpha\beta = \beta\alpha$ then $\alpha\beta$ also has the same component equal to zero.

Proof : The proof is done by taking the second component to be equal to zero. For the third and the fourth component the proof will follow similarly. Let $\alpha = a+\phi i+cj+dk$ and $\beta = a'+\phi i+c'j+d'k$. So

$$\alpha\beta = (aa' + cc' - dd') + (cd' - c'd)i + (ac' + a'c)j + (ad' + a'd)k$$
$$\beta\alpha = (aa' - cc' - dd') + (c'd - cd')i + (ac' + a'c)j + (ad' + a'd)k$$

If $\alpha\beta = \beta\alpha$ then $cd' - c'd = c'd - cd' \Rightarrow cd' - c'd = 0$ i.e. $\alpha\beta$ has its second component equal to zero. □

Corollary 2.1 — If α be a quaternion whose one component (except the first) is zero then α^n also has the same component equal to zero for some $n \in \mathbb{Z}^+$ (the set of all positive integers).

PROOF : The proof is by induction on n. The result trivially holds for $n = 1$.

Let α^m has the same component equal to zero which is zero in α. Now $\alpha^m\alpha = \alpha\alpha^m \Rightarrow \alpha\alpha^m$ has the same component equal to zero (by theorem 2.1) i.e. α^{m+1} has the same component equal to zero. This proves the corollary. □

The result is true for negative integers also. One can have $\alpha^{-1} = \bar{\alpha}/N(\alpha)$. So if α have some component equal to zero then the same component is equal to zero in $\bar{\alpha}$ and hence in α^{-1}. So the following corollary follows.

Corollary 2.2 — If α be a quaternion with one component zero (except the first) then the same component is zero in α^n for $n \in \mathbb{Z}$.

3. CONJUGATE PAIR OF TETRAHEDRA

Let (A), (B) and (C) be three tetrahedra where the faces of (C) are the polars of the vertices of (B) for (A). But this does not imply that the polars of the vertices of (C) are the faces of (B). So the following definition comes into picture.

Definition 3.1 — (B) and (C) are called the conjugate for (A) if B_i and C_k $(i \neq k)$ are conjugate points $(i, k \in \psi_3)$ for (A).

Hence if $(B)_i = ((b_{ij}))$ and $(C)_i = ((c_{ij}))$ then from (1.8) the conditions for conjugacy are

$$\left.\begin{aligned}((b_{ij}))((c_{ij}))^* \big]^{-1} &= \text{diag}\,(m_0, m_1, m_2, m_3),\ m_0 m_1 m_2 m_3 \neq 0\\ ((c_{ij}))((b_{ij}))^* \big]^{-1} &= \text{diag}\,(n_0, n_1, n_2, n_3),\ n_0 n_1 n_2 n_3 \neq 0\end{aligned}\right\} \quad \ldots (3.1)$$

Given a tetrahedron (B) it is not always possible to find (C) such that (B) and (C) will be conjugate for a third tetrahedron. In general given (B) there does not exist conjugate of (B) for (A). But given a point there can be constructed a pair of tetrahedra with the given point as the vertex such that the pair forms a conjugate pair for (A).

Let X be any point not on the face of (A) whose coordinates for (A) are (p, q, r, s), $pqrs \neq 0$. So there is a tetrahedron (B) such that $(B)_1 = \alpha = p + qi + rj + sk$ where $B_1 = X$ and , the vertices of (B) are not on the faces of (A). Let

$\alpha^* = p^{-1} + q^{-1}i + r^{-1}j + s^{-1}k$. Let (C) be another tetrahedron such that $(C)_A = \beta, \beta$ is a quaternion. Then by (1.8), (C) is the polar tetrahedron of (B) if and only if

$$\beta\bar{\alpha}^* = t(\neq 0) \in \mathcal{F}$$

(note that the conjugate of (α^*) is $\bar{\alpha}^*$). Multiplying β to the above expression one gets $\bar{\beta}\beta\bar{\alpha}^* = t\bar{\beta} \Rightarrow t\bar{\beta} = \bar{\alpha}^* \Rightarrow \beta = t^{-1}\alpha^*$ or in our notation $\beta \equiv \alpha^*$. In that case $\beta^* \equiv \alpha \Rightarrow \alpha\bar{\beta}^* \equiv \alpha(\bar{\alpha}^*) \equiv \alpha\bar{\alpha} \in \mathcal{F}$. So (B) is the polar tetrahedron of (C) and (A). Thus the following theorem follows.

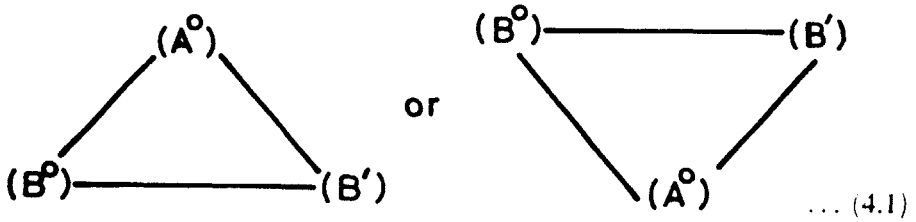
Theorem 3.1—If $(B)_A = \alpha = p + qi + rj + sk$, $pqrs \neq 0$, and $(C)_A = \beta$ then (B) and (C) form a conjugate pair of (A) if and only if $\beta \equiv \alpha^*$.

It is known from the definition of a conjugate pair that a tetrahedron cannot have a conjugate for (A) if one of its vertices is on the face of (A). In order to have a corresponding β all the components of α has to be non-zero. Any quaternion all of whose components are non-zero represents a tetrahedron possessing a conjugate counterpart with respect to (A).

Note 2: As $a + bi + cj + dk \equiv 1 + a^{-1}(bi + cj + dk)$, the conjugate pair of tetrahedra described in this paper is called a three parametric conjugate pair. There are also four parametric conjugate pair. But it is not possible to study them using quaternions and hence is not covered here.

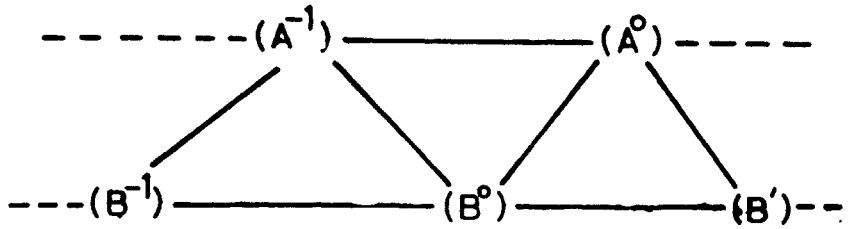
4. Conjugate Systems

In this article (X^n) is a tetrahedron whose vertices are X_0^n, X_1^n, X_2^n and X_3^n, n is an integer. Let (A^0) and (B^0) be two tetrahedra such that $(B^0)A^0 = \alpha = p + qi + rj + sk$ where $pqrs \neq 0$. So the conjugate of (B^0) for (A^0) exists and is denoted by (B^1) where $(B^0)A^0 = \beta = p^{-1} + q^{-1}i + r^{-1}j + s^{-1}k$. Again $(A^0)B^0 = \bar{\alpha} = p - qi - rj - sk$ implies that the conjugate of (A^0) for (B^0) which is denoted by (A^{-1}). Also exists the conjugate of (A^0) for (B^1) which is denoted by (A^1). Similarly there exist the conjugate of (B^0) for (A^{-1}) and of (B^1) for (A^1) and is denoted as (B^{-1}) and (B^2) respectively. In this way an infinite (depends upon the field $\mathcal{F}$) system of conjugate tetrahedra is generated. The conjugate pair $(B^0), (B^1)$ for (A^0) can be expressed diagramatically as



... (4.1)

where the tetrahedra on the horizontal edge of any triangle are conjugate for the tetrahedron at the opposite vertex. This can be extended in the following way



... (4.2)

where given a triangle the tetrahedra at the horizontal edge are conjugate for the tetrahedron at the opposite vertex.

In the chain (4.2) both the rows can be extended to infinite number of tetrahedra (if the field is infinite). Here for any integer m, (B^m) and (B^{m+1}) are the conjugate pairs of (A^m). Also (A^m) and (A^{m+1}) are the conjugate pairs for (B^{m+1}). The upper row is called the first system of conjugate tetrahedra and is denoted by $CS(1)$ and the lower is called the second system and is denoted by $CS(2)$. $CS(1)$ and $CS(2)$ are the complement to each other. Both the systems exist or do not exist simultaneously.

Given $(B^0)A^0 = \alpha = p + qi + rj + sk$, $pqrs \neq 0$ the following can be obtained.

$$\begin{aligned}(B^1)A^0 = \beta &= p^{-1} + q^{-1}i + r^{-1}j + s^{-1}k \\ &\equiv qrs + prsi + pqsj + pqrk \\ (B^1)B^0 &= (B^1)A^0 \cdot (A^0)B^0 = \beta\bar{\alpha}. \qquad \ldots (4.3)\\ (A^0)_{B^1} = \bar{\beta} &\Rightarrow (A^1)_{B^1} = \bar{\alpha} \Rightarrow (B^1)_{A^1} = \alpha \Rightarrow (B^2)_{A^1} = \beta \\ \Rightarrow (A^1)_{B^2} = \bar{\beta} &\Rightarrow (A^2)_{B^2} = \bar{\alpha} \Rightarrow (B^2)_{A^2} = \alpha.\end{aligned}$$

Giving a different notation the following can be obtained.

$$\begin{aligned}(B^{-1})_{B^0} &= \beta\alpha = \bar{\mu} \\ (B^2)_{B^0} &= (B^2)_{A^1} \cdot (A^1)_{B^1} \cdot (B^1)_{B^0} = \beta\bar{\alpha}\mu = \mu\mu = \mu^2 \qquad \ldots (4.4)\\ (A^1)_{A^0} &= (A^1)_{B^1} \cdot (B^1)_{A^0} = \bar{\alpha}\beta = \nu \\ (A^2)_{A^0} &= (A^2)_{B^0} \cdot (B^2)_{A^1} \cdot (A^1)_{A^0} = \bar{\alpha}\beta\nu = \nu^2\end{aligned}$$

Theorem 4.1 — For any positive integer n, $(B^n)_{B^0} = \mu^n$ and $(A^n)_{A^0} = \nu^n$.

where $(B^0)_{A^0} = \alpha$, $(B^1)_{A^0} = \beta$ and $\mu = \beta\bar{\alpha}$ and $\bar{\alpha}\beta = \nu$.

PROOF: The proof is done by induction n. From (4.4) it is known that the result is true for $n = 1$ (even for $n = 2$). Let the result be true for $n = m$. So $(B^m)_{B^0} = \mu^m$ and $(A^m)_{A^0} = \nu^m$. Now

$$\begin{aligned}(A^m)_{A^0} = \nu^m &= (\bar{\alpha}\beta)^m = \bar{\alpha}(\beta\bar{\alpha})^{m-1}\beta = \bar{\alpha}\mu^{m-1}\beta \Rightarrow \\ (A^0)_{A^m} &= \bar{\beta}\mu^{1-m}\alpha \qquad \ldots (4.5)\end{aligned}$$

Therefore $(B^m)_{A^m} = (B^m)_{B^0}(B^0)_{A^0}(A^0)_{A^m} = \mu^m \bar{\alpha}\beta\mu^{1-m}\alpha$. So

$$(B^m)_{A^m} = \alpha \qquad \ldots (4.6)$$

and $(B^{m+1})_{A^m} = \beta \qquad \Rightarrow (A^m)_{B^{m+1}} = \bar{\beta} \qquad \ldots (4.7)$

Hence $(A^{m+1})_{B^{m+1}} = \bar{\alpha} \qquad \ldots (4.8)$

So $(B^{m+1})_{B^0} = (B^{m+1})_{A^m}(A^m)_{A^0}(A^0)_{B^0} = \beta\bar{\alpha}\mu^{m-1}\beta\bar{\alpha}$

$$= \mu\mu^{m-1}\mu = \mu^{m+1} \qquad \ldots (4.9)$$

And $(A^{m+1})_{A^0} = (A^{m+1})_{B^{m+1}}(B^{m+1})_{B^0}(B^0)_{A^0} = \bar{\alpha}\mu^{m+1}\alpha$

$$= \bar{\alpha}\beta\nu^m\bar{\alpha}\alpha = \nu\nu^m = \nu^{m+1} \qquad \ldots (4.10)$$

From (4.9) and (4.10) it is concluded that the result is true for $n = m + 1$ if it is true for $n = m$. Hence the theorem is proved.

Instead of (B^0) one could have started with any tetrahedron of the chain. Infact if one start with (B^n) then $(B^{-n})_{B^0} = \mu^{-n}$. Hence $(B^n)_{B^{-n}} = \mu^{2n}$. So the following corollary is clear.

Corollary 4.1 — $(B^i)_{B^j} = \mu^{j-i}$ and $(A^i)_{A^j} = \nu^{j-i}$ for $i, j \in Z$.

5. Mobius And Desmic Conjugate Systems

From article

$$-4(B^1)_{B^0} = \mu = \beta\alpha = 4pqrs + (pr - qs)(ps + qr)i + (ps - qr)(pq + rs)j + (pq - rs)(pr + qs)k \qquad \ldots (5.1)$$

As $pqrs \neq 0$ the first component of μ cannot be zero. So besides forming a conjugate pair for (B^0), (A^0) and (B^1) form a mobius pair if one component of μ (except the first component) is zero (by 2.3). Which implies that μ^m also has that particular component equal to zero (by corollary 2.2). So any two tetrahedra of $CS(2)$ form a Mobius pair if (B^0) and (B^1) form a Mobius pair. This gives the following result.

Theorem 5.1 — If (B^0) and (B^1) of $CS(2)$ form a Mobius pair then any two tetrahedra of $CS(2)$ will also do.

Again

$$(A^1)A^0 = \nu = \bar{\alpha}\beta = 4pqrs + (pr + qs)(ps - qr)i + (ps + qr)(pq - rs)j + (pq + rs)(pr - qs)k. \qquad \ldots (5.2)$$

If (B^0) and (B^1) are Mobius then μ has one of its components (except the first) equal to zero. From (5.1) it is obtained that (B^0) and (B^1) are Mobius if one of the six factors

$(pr - qs), (ps + qr), (ps - qr), (pq + rs), (pq - rs)$ and $(pr + qs)$ is zero. If one of the above six factors are zero then in (5.2) one can see that ν also has one component equal to zero. So (A^0) and (A^1) are Mobius pair. Applying Theorem 5.1 the following result is obtained.

Theorem 5.2—If (B^0) and (B^1) of $CS(2)$ form a Mobius pair then any two tetrahedra of $CS(1)$ also form a Mobius pair.

Theorem 5.3—If (B^0) and (B^1) form a desmic pair then $Cs(2)$ is reduced to a finite system having three tetrahedra which form a desmic system.

Note 3 : If one of the conjugate systems is reduced to a finite system having n number of tetrahedra then the other system also becomes finite having the same number of tetrahedra. Because if $\mu^n \in \mathcal{F}$ then $\nu^n \in \mathcal{F}$ [This is possible over all real field. For example the quaternion $a + bi + cj + dk$ with the condition $a^2 = 3(b^2 + c^2 + d^2)$ has its sixth power in $\mathcal{F}$]. This can be obtained by the followings.

$$\mu^n \in \mathcal{F} \Rightarrow \mu^{n-1} \equiv \bar{\mu} \Rightarrow \nu^n = \bar{\alpha}(\beta\bar{\alpha})^{n-1}\beta$$
$$= \bar{\alpha}\bar{\mu}^{n-1}\beta \equiv \bar{\alpha}(\bar{\beta}\bar{\alpha})\beta = \bar{\alpha}\alpha\bar{\beta}\beta \in \mathcal{F}$$

Similarly $\nu^n \in \mathcal{F} \Rightarrow \mu^n \in \mathcal{F}$.

Proof : If $(B^1)_{B^0} = \mu$ then μ can be written a simpler form as

$$\mu = a + bi + cj + dk \qquad \text{where}$$
$$a = 4pqrs,\ b = (pr - qs)(ps + qr),$$
$$c = (ps - qr)(pq + rs),\ d = (pq - rs)(pr + qs).$$

If (B^0) and (B^1) are desmic then from (2.2)

$$a^2 = b^2 = c^2 = d^2. \text{ So } \mu^2 = (a^2 - b^2 - c^2 - d^2) + 2abi + 2acj + 2adk$$
$$= -2a^2 + 2abi + 2acj + 2adk$$
$$\equiv a - bi - cj - dk = \bar{\mu} \qquad \ldots (5.3)$$

This implies $\mu^3 \in \mathcal{F}$. So when (B^0) and (B^1) are desmic then $CS(2)$ is reduced to a finite system having only (B^{-1}), (B^0) and (B^1) are desmic. Hence $CS(2)$ is reduced to a desmic system.

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