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ON STRONG APPROXIMATION TO A FUNCTION BY ITS JACOBI SERIES

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§1. *Definition* —An infinite series ΣU_n is said to be strongly summable with index $k > 0$ to the sum S if

$$\sum_{\nu=0}^{n} | S_\nu - S |^k = O(n), \; n \to \infty$$

where S_ν denotes the νth partial sum of the series ΣU_n.

Let $f(x)$ be a function defined on the interval $-1 \leqslant x \leqslant 1$ such that the integral.

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)\, dx$$

exists in the sense of Lebesgue. The Fourier-Jacobi expansion of $f(x)$ is given by,

$$f(x) \sim \sum_{n=0}^{\infty} a_n \, p_n^{(\alpha,\beta)}(x) \qquad \ldots(1.1)$$

where $p_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of order (α, β) and degree n in x and,

$$a_n = \frac{1}{g_n} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \; p_n^{(\alpha,\beta)}(x)$$

and

$$g_n = \frac{2^{\alpha+\beta}\,1}{2n+\alpha+\beta} \; \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \; \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}.$$

We write

$$\phi(\omega) = f(\cos\omega) - f(1).$$

§2. In a recent paper Sulaiman[2] has proved the following :

Theorem —If $\alpha > -1$, $\beta > -1$, $\alpha + \beta > 0$, and if $\psi(t)$ is a positive increasing function such that,

$$\int_0^t | d\phi(u) | du \leqslant A\psi(t), \ A \text{ being constant} \qquad \ldots(2.1)$$

$$\int_t^\delta u^{\alpha-1/2}\psi(u)\,du = O\{t^{\alpha+1/2}\psi(t)\} \qquad \ldots(2.2)$$

and

$$\phi(t) = \int_{t\to 0}^{a} u^{\beta+1/2}\phi(\pi-u)\,du = O\{t^{\beta+1/2}\psi(t)\} \qquad \ldots(2.3)$$

then

$$\frac{1}{n+1}\sum_{\nu=0}^{n} | S_\nu(1) - f(1) | = O\{\psi(1/n)\}.$$

Here we observe that conditions (2.1) and (2.2) may be replaced by a more general condition. In fact, we see that

$$| \phi(t) | = | \int_0^t d\phi(u)\,du \leqslant \int_0^t | d\phi(u) | du \leqslant A\psi(t)$$

$$\Rightarrow | \phi(t) | \leqslant A\psi(t).$$

Then from (2.2) of Sulaiman we get

$$\int_t^\delta u^{\alpha-1/2} | \phi(u) | du = O(t^{\alpha+1/2}(t)). \qquad \ldots(2.4)$$

The object of the present paper is to prove a similar theorem by replacing (2.1) and (2.2) by a more general condition (2.4). In fact, we prove the following :

Theorem —If $\alpha > -1$, $\beta > -1$, $\alpha + \beta > 0$ and if $\psi(t)$ is a positive increasing function which along with (2.4) satisfies

$$\phi(t) = \int_{t0}^{a} u^{\beta+1/2}\phi(\pi-u)\,du = O(t^{\alpha+1\cdot 2}\psi(t)) \qquad \ldots(2.5)$$

then

$$\frac{1}{n+1}\sum_{\nu=0}^{n} | S_\nu(1) - f(1) | = O(\psi(1/n)).$$

§ 3. To prove the theorem we require following lemmas :

Lemma 1 (Szegö[3])—Let α, β be arbitrary real numbers and C a fixed positive constant then, for $n \to \infty$

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2}\, O(n^{-1/2}), & c/n \leqslant \theta \leqslant \pi/2 \\ O(n^{\alpha}), & 0 \leqslant \theta \leqslant c/n. \end{cases}$$

Lemma 2 (Szego[2])—If $\alpha > -1$, $\beta > -1$, then

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2} K(\theta) \{\cos(N\theta + \nu) + (n \sin\theta)^{-1} O(1)\}$$

where $c/n \leqslant \theta \leqslant \pi - c/n$

$$K(\theta) = \pi^{-1/2} (\sin\theta/2)^{-\alpha-1/2} (\cos\theta/2)^{-\beta-1/2}$$

$$N = n + \tfrac{1}{2}(\alpha + \beta + 1), \nu = -(\alpha + \tfrac{1}{2}).\ \pi/2.$$

Lemma 4—The condition (2.3) implies

$$\int_0^t \omega^{2\beta} |\phi(\pi - \omega)|\, d\omega = O(t^{\alpha+\beta}\, \psi(t)).$$

Proof of lemma 3 is fairly straight forward and we omit the details.

Lemma 4—Under the condition (2,4) we easily get

$$\int_0^t |\phi(u)|\, du = O(t\, \psi(t)).$$

§4. *Proof of the theorem* —Let S_n be the nth partial sum of the series (1.1). Hence we have

$$S_n(1) = \int_0^{\pi} (1 - \cos\omega)^{\alpha+1/2} (1 + \cos\omega)^{\beta+1/2} f(\cos\omega). \sum_{m=0}^{n} (1/gm) \times P_m^{(\alpha,\beta)}(\cos\omega)\, P_m^{(\alpha,\beta)}(1)\, d\omega$$

then

$$S_n(1) - f(1) = \int_0^{\pi} (\sin\omega/2)^{2\alpha+1} (\cos\omega/2)^{2\beta+1} \phi(\omega) \times \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1)\, \Gamma(n + \beta + 1)} P_n^{(\alpha+1,\beta)}(\cos\omega)\, d\omega$$

$$= \frac{\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1)\, \Gamma(n + \beta \mp 1} \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi - 1/n} + \int_{\pi-1/n}^{\pi} \right]$$

(*equation continued on p.* 170)

$$\equiv J_1 + J_2 + J_3 + J_4, \text{ say}$$

$$J_1 = O(n^{2\alpha+2}) \int_0^{1/n} \omega^{2\alpha+1} \mid \phi(\omega) \mid d\omega$$

$$= O(\psi(1/n)) \text{ by Lemma 4} \qquad \ldots(4.1)$$

$$J_2 = O(n^{\alpha+1}) \int_{1/n}^{\delta} (\sin \omega/2)^{2\alpha+1} (\cos \omega/2)^{2\beta+1} \mid \phi(\omega) \mid .$$

$$\times [O(n^{-1/2}) (\sin \omega/2)^{-\alpha-3/2} (\cos \omega/2)^{-\beta-1/2} (O(1) + (n \sin \omega)^{-1})\}]$$

$$= O(n^{\alpha+1/2}) \int_{1/n}^{\delta} \omega^{\alpha-1/2} \mid \phi(\omega) \mid d\omega + O(n^{\alpha-1/2}) \cdot \int_{1/n}^{\delta} \omega^{\alpha-3/2} \mid \phi(\omega) \mid d\omega$$

$$= O(n^{\alpha+1)2} \cdot \frac{1}{n^{\alpha+1|2}} \psi(1/n)) \text{ by (2.4)}$$

$$= O(\psi(1/n)) \qquad \ldots(4.2)$$

$$J_3 = O(n^{\alpha+1/2}) \int_{\delta}^{\pi-1/n} (\sin \omega/2)^{\alpha-1/2} (\cos \omega/2)^{\beta-1/2} \cdot \mid \phi(\omega) \mid d\omega$$

$$= O(\psi(1/n)) \text{ by (2.5)} \qquad \ldots(4.3)$$

$$J_4 = \lambda_n \int_{\pi-1/n}^{\pi} (\sin \omega/2)^{2\alpha+1} (\cos \omega/2)^{2\beta+1} \mid \phi(\omega) \mid . \; P_n^{(\alpha+1,\beta)} (\cos \omega) \, d\omega$$

where

$$\lambda_n = \frac{(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(n + \beta + 1)}$$

$$= O(\psi(1/n)) \text{ by Lemma 3.} \qquad \ldots(4.4)$$

In view of (4.1), (4.2), (4.3) and (4.4) we get

$$\mid S_n(1) - f(1) \mid = O\{\psi(1/n)\}.$$

Therefore

$$\frac{1}{(n+1)} \sum_{\nu=0}^{n} \mid S_\nu(1) - f(1) \mid = O\{\psi\, 1/n)\}.$$

This completes the proof of the theorem.

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