Indian J. pure appl. Math., 10(2) : 204-208, February 1979

INTRINSIC EQUATIONS FOR THE MAGNETOFLUID FLOWS

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(*Received* 15 *February* 1978; *after revision* 25 *November* 1978)

The paper presents the intrinsic equations governing the behaviour of the magnetofluid flows employing Bonnet-Kovalavsky formulae for the congruences of the stream lines and the magnetic field lines. Some theorems have also been established in case of the transverse flows.

1. INTRODUCTION

The geometry of non-conducting fluids flows has been of interest for a long time, but that of magnetofluid flows has had little attention. Wasserman (1967) has proved that the Faraday's equation for a steady magnetofluid ensures the existence of the family of surfaces formed by the flow and field lines, which essentially determine everything else about the flows. Suryanarayan (1972) has obtained certain intrinsic equations for the magnetofluid flows and derived a class of helicoidal equations.

The purpose of this paper is to obtain certain intrinsic equations for the steady and non-dissipative magnetofluid flows and to establish some theorems for the transverse flows. We have studied some geometric properties of the congruences of the stream lines and the magnetic field lines.

2. FIELD EQUATIONS

Let x^i $(i = 1, 2, 3)$ denote the variables of a system of Cartesian orthogonal coordinates. In order to use the summation convention we shall use the indices in covariant and contravariant positions. The comma will denote the partial differentiation throughout this paper. The equation of continuity, the equation of motion, and the Maxwell equations for a steady non-dissipative magnetofluid flows are (Suryanarayan 1972)

$$(\rho u^i)_{,i} = 0 \qquad \ldots(2.1)$$

$$\rho u^j u_{k,j} + p_{,k} = h^j h_{k,j} - \tfrac{1}{2} (h^2)_{,k} \qquad \ldots(2.2)$$

$$u^i S_{,i} = 0 \qquad \ldots(2.3)$$

$$(u^i h^j - u^j h^i)_{,j} = 0 \qquad \ldots(2.4)$$

$$h^j_{,j} = 0 \qquad \ldots(2.5)$$

and

$$\rho = f(p, S) \qquad ...(2.6)$$

where p, ρ, S and u^i are the fluid pressure, fluid density, entropy of the fluid and the velocity vector respectively. $h_i = \sqrt{\mu}\, H_i$, where μ and H_i are the magnetic permeability and magnetic field vector respectively and $h^i h_i = h^2$.

3. Intrinsic Equations

The Faraday's eqn. (2.4) gives the family of surfaces $\varphi =$ constant containing the flow and field lines; that is (Suryanarayan 1972)

$$\epsilon_{ijk} h^j u^k = \varphi_{,i}. \qquad ...(3.1)$$

Let (t^i, n^i, b^i); (T^i, N^i, b^i) are the triads along the stream lines and magnetic field lines respectively. If α be the angle between the stream lines, $u^i = ut^i$, and the field lines, $h^i = hT^i$, on the surface φ, then we may write

$$t^i = T^i \cos\alpha - N^i \sin\alpha \qquad ...(3.2)$$

$$n^i = T^i \sin\alpha + N^i \cos\alpha \qquad ...(3.3)$$

$$T^i = t^i \cos\alpha + n^i \sin\alpha \qquad ...(3.4)$$

$$N^i = n^i \cos\alpha - t^i \sin\alpha. \qquad ...(3.5)$$

The Bonnet-Kavalevsky formulae (see Goettz 1970) for the congruence of the stream lines are

$$\begin{pmatrix} Dt^i \\ Dn^i \\ db^i \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} t^i \\ n^i \\ b^i \end{pmatrix} \qquad ...(3.6)$$

where D denotes the absolute differentiation along the stream lines. k_g, k_n, τ_g are the geodesic curvature, the normal curvature and the geodesic torsion of the stream lines respectively.

The Bonnet-Kovalevsky formulae may be written for the magnetic field lines as follows:

$$\begin{pmatrix} D^*T^i \\ D^*N^i \\ D^*b^i \end{pmatrix} = \begin{pmatrix} 0 & k_g^* & k_n^* \\ -k_g^* & 0 & \tau_g^* \\ -k_n^* & -\tau_g^* & 0 \end{pmatrix} \begin{pmatrix} T^i \\ N^i \\ b^i \end{pmatrix} \qquad ...(3.7)$$

where D^* denotes the absolute differentiation along the magnetic field lines. k_g^*, k_n^*, τ_g^* are the geodesic curvature, the normal curvature and the geodesic torsion of the field lines respectively.

Using (2.1) and (2.5) in (2.4), we get

$$(\ln \rho)_{,j} u^j h^k - u^j h^k_{,j} + h^j u^k_{,j} = 0 \qquad \ldots(3.8)$$

which on multiplication by t_k reduces to

$$(\ln \rho)_{,j} u^j h_{t\,|} - u^j t^k h_{k,j} + t^k h^j u_{k,j} = 0 \qquad \ldots(3.9)$$

where $h_{t\,|}$ denotes the magnetic intensity along the stream lines. Using (3.4) and (3.6) in (3.9), we have

$$hD^*(\ln u) + h_{t\,|}\, D(\ln \rho) - D(h_{t\,|}) + hk_g \sin \alpha = 0. \qquad \ldots(3.10)$$

Similarly multiplying (3.8) by T_k, n_k, b_k and N_k and using (3.2) - (3.7), we obtain

$$D(\ln \rho - h) + hk^*_g \sin \alpha - h/uD^*(u_{T\,|}) = 0 \qquad \ldots(3.11)$$

$$h_{n\,|}\, D(\ln h_{n\,|}/\rho) + h(k_g \cos \alpha + D^*\alpha - k^*_g) = 0 \qquad \ldots(3.12)$$

$$\tan \alpha = (k^*_n - k_n)/(\tau_g + \tau^*_g) \qquad \ldots(3.13)$$

and

$$D^*(u_{N\,|}) - u(k_g + D\alpha - k^*_g \cos \alpha) = 0 \qquad \ldots(3.14)$$

respectively. Here $h_{n\,|}$ denotes the normal component of the magnetic field along the stream line. $u_{T\,|}$, $u_{N\,|}$ are the tangential and normal components of the velocity along the field lines respectively.

Multiplying (2.2) by t_k, n_k, T_k, N_k and b_k, and using (3.2) - (3.7), we get

$$\rho D(\ln u) + D(p + \tfrac{1}{2}h^2) - hD^*(h_{t\,|}) - h^2 \sin^2 \alpha\, D^*\alpha + h^2 k^*_g \sin \alpha = 0 \qquad \ldots(3.15)$$

$$\rho u^2 k_g + \frac{d}{dn}(p + \tfrac{1}{2}h^2) - hD^*(h_{n\,|}) + h^2 \cos \alpha\, D^*\alpha - h^2 k^*_g \cos \alpha = 0 \qquad \ldots(3.16)$$

$$D^*p + \rho u D(u_{T\,|}) + \rho u^2 \sin \alpha\, D\alpha - \rho u^2 k_g \sin \alpha = 0 \qquad \ldots(3.17)$$

$$\rho u D(u_{N\,|}) + \rho u^2 \cos \alpha (D\alpha + k_g) + \frac{d}{dN}(p + \tfrac{1}{2}h^2) - h^2 k^*_g = 0 \qquad \ldots(3.18)$$

and

$$\frac{d}{db}(p + \tfrac{1}{2}h^2) - (h^2 k^*_n - \rho u^2 k_n) = 0 \qquad \ldots(3.19)$$

respectively. Equations (3.10) - (3.19) are the intrinsic equations governing the general behaviour of the magnetofluid flows.

Theorem 3.1 — The vorticity of the magnetofluid vanishes iff the magnitude of the velocity and the fluid pressure are constant along the magnetic field lines.

PROOF : Equation (2.2) can be written as

$$p_{,k} = h^j(h_{k,j} - h_{j,k}) - \rho u^j u_{k,j}. \qquad \text{...(3.20)}$$

Multiplying (3.20) by T_k and using (3.4) and the definitions $-J^i = \epsilon^{ijk} h_{j,k}$; $w^i = \epsilon^{ijk} u_{j,k}$, where J^i, w^i are the current vector and the vorticity vector respectively, we get

$$D^*p + \rho u\, D^*u = -\rho u \sin\alpha\, w^k b_k,\ \alpha \neq 0 \qquad \text{...(3.21)}$$

which proves the statement.

4. TRANSVERSE FLOW

In this section we will consider the case in which the magnetic field lines and the stream lines are orthogonal to each other and establish some theorems of interest.

Theorem 4.1 — The hydromagnetic pressure is constant along the normals to the stream lines iff the mechanical energy and density are constant along the magnetic field lines.

PROOF : By virtue of (3.14) and (3.16), we get

$$\frac{d}{dn}(p + \tfrac{1}{2}h^2) = D^*(\tfrac{1}{2}h^2 + \tfrac{1}{2}\rho u^2) - \tfrac{1}{2}u^2 D^*\rho \qquad \text{...(4.1)}$$

which proves the statement when $\alpha = \frac{1}{2}\pi$.

Theorem 4.2 — If the density is constant along the stream lines, the geodesic curvature of the magnetic field lines is given by

$$k^*_g = J^k b_k/2h.$$

PROOF : We observe the relation

$$h^i u_i = hu \cos\alpha \qquad \text{...(4.2)}$$

which yields

$$h_{i,j} u^i T^j + h u_{i,j} T^i T^j = u_{T\,|} D^*h + h_{t\,|} D^*u - h_{n\,|} uD^*\alpha. \qquad \text{...(4.3)}$$

Multiplying (2.4) by T_k and using (2.1), (4.3) and the definition of the current vector, we get

$$huD(\ln\rho) + u_{N\,|}(J^k b_k - 2hk^*_g) + 2hD^*u_{T\,|}$$

$$- u_{T\,|} D^*h - h_{t\,|} D^*u + h_{n\,|} uD^*\alpha = 0 \qquad \text{...(4.4)}$$

which proves the statement when $\alpha = \frac{1}{2}\pi$.

Theorem 4.3 — The magnitude of velocity is constant along the magnetic field lines iff the stream lines are the geodesics.

The proof of the theorem follows from (3.10).

Theorem 4.4 — The density of the fluid is balanced by the magnetic intensity along the stream lines iff the magnetic field lines are the geodesics.

The proof of the theorem is obvious from (3.11).

Remark : The transverse flow satisfies the relation

$$\tau_g + \tau_g^* = 0, k_n^* \neq k_n.$$

REFERENCES

Goettz, Abraham (1970). Introduction to Differential Geometry. Addison Wesley Publishing Company, London, p. 229.

Suryanarayan, E. R. (1972). Intrinsic equations for hydromagnetic flows. *Rev. Roum. Math. Pures Appl.*, **17**, 103.

Wasserman, R. H. (1967). On the geometry of magnetohydrodynamic flows. *Q. Jl Mech. appl. Math.*, **20**, 219.