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ON A PAPER OF PHADKE, KHASBARDAR AND THAKARE

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An operator $T \in \beta(H)$ is said to be quasi-unitary if $TT^* = T^*T = T + T^*$ or equivalently $T = I + U$, where U is unitary. Using the fact that for a QU operator $T, \sigma(T) \subset \{\lambda \mid |\lambda - 1| = 1\}$ some of the earlier results on QU operators are simplified. Some spectral sets are given for such operators and we further characterise QU operators which are unitary or self-adjoint. Also we prove that if T and T^2 are QU, then T is the zero operator.

Recently, Phadke *et al.* (1977) have introduced the notion of a quasi-unitary (QU) operator on a Hilbert space.

Definition — An operator $T \in \beta(H)$ is said to be QU iff $T^*T = TT^* = T + T^*$. Thus a QU operator is essentially normal. It is easy to verify that T is QU iff $T - I$ is unitary (*see* Result 3.1 of Phadke *et al.* 1977). West (1965) has proved that a compact normal operator has its spectrum contained in C_1, the circle with centre 1 and radius 1 iff $T^*T = T + T^*$. If $T \in \beta(H)$ is QU, then $T = I + U$, where U is unitary and hence $\sigma(T) \subset \{\lambda \mid |\lambda - 1| = 1\} = C_1$.

Thus if T is a QU operator, $T + I$ is invertible and any complex number μ with $|\mu - 1| < 1$ is a regular value for T. This combined with the above characterization of QU operators enables us to give simpler proofs of some of the results proved by Phadke *et al.* (1977). We give some additional properties of QU operators.

Notations: For an operator $T \in \beta(H)$ we denote by $\sigma(T)$, $\pi_0(T)$ and $W(T)$ respectively the spectrum, the point spectrum and the numerical range of T. $R_\lambda(T)$ denotes $(T - \lambda I)^{-1}$ whenever it exists. If $A, B \in \beta(H)$ we write $A \leftrightarrow B$ if $AB = BA$.

Phadke *et al.* (1977) have proved that if T is QU and unitary, T is invertible and T^{-1} is also QU. The following theorem characterizes such an operator.

Theorem 1 — If T is QU and unitary, $\sigma(T) \subset \{p, \bar{p}\}$ where $p = \frac{1}{2} + i\sqrt{\frac{3}{2}}$. In this case T is either a scalar multiple of identity or the direct sum of two scalar multiples of identity operator.

PROOF : Since T is QU and unitary,

$$\sigma(T) \subset C_1 \cap \{\lambda \mid |\lambda| = 1\} \Rightarrow \sigma(T) \subset \{p, \bar{p}\}.$$

If p or $\bar{p} \in \sigma(T)$, it is an isolated point and hence $\in \pi_0(T)$. Thus if $\sigma(T) = \{p\}$ or $\{\bar{p}\}$, $T = pI$ or $\bar{p}I$. If $\sigma(T) = \{p, \bar{p}\}$, $T = pI \oplus \bar{p}I$.

The following theorem characterizes self-adjoint QU operators.

Theorem 2 — If T is QU and self-adjoint, then either $T = 0$ or $T = 2I$ or T is the direct sum of the zero operator and $2I$.

PROOF : Since T is self-adjoint and QU,

$$\sigma(T) \subset C_1 \cap \{\lambda \mid \lambda \text{ is real}\} \Rightarrow \sigma(T) \subset \{0, 2\}.$$

Thus $T = 0$ if $\sigma(T) = \{0\}$, $T = 2I$ if $\sigma(T) = \{2\}$ and finally $T = 0 \oplus 2I$ if $\sigma(T) = \{0, 2\}$.

Corollary 1 — The only non-zero projection operator which is QU is $2I$.

Phadke *et al.* (1977) have proved that $\{\lambda \mid \text{Re}\, \lambda \geqslant 0\}$ is a spectral set for T. We give a simple proof of this result and improve it as follows :

Theorem 3 — If T is a QU operator, the following sets are spectral sets for T:

(I) $\{\lambda \mid \mid \lambda - 1 \mid \leqslant 1\}$

(II) $\{\lambda \mid \text{Re}\, \lambda \geqslant 0\}$

(III) $\{\lambda \mid \mid \lambda + \mu \mid \geqslant \mu, \text{ where } > 0\}$.

To prove the theorem, we need the following results.

Lemma 1 — If T is a QU operator then Re $W(T) \geqslant 0$.

PROOF : If $\lambda \in W(T)$ then $\lambda = (Tx, x)$ for some unit vector $x \in H$.

$\therefore$ $\quad$ $\text{Re}\, \lambda = ((T + T^*)x, x) = (T^*Tx, x) = \| Tx \|^2 \geqslant 0.$

Lemma 2 — If T is a QU operator, $\| R_{-\lambda}(T) \| \leqslant 1/\lambda \;\forall\; \lambda > 0$. In particular, $\| (T + I)^{-1} \| \leqslant 1$.

PROOF : By lemma 1, for the operator $-T$, Re $\overline{W(-T)} \leqslant 0$. Hence by a lemma due to Orland (1964) for $\lambda > 0$,

$$\| (-T - \lambda I)^{-1} \| = \| (T + \lambda I)^{-} \| \leqslant 1/\lambda.$$

Lemma 3 (Saitô 1972) — The closed sets

$$\{\lambda \mid \mid \lambda - \alpha \mid \leqslant r\}, \{\lambda \mid \text{Re}\, \lambda \geqslant 0\} \text{ and } \{\lambda \mid \mid \lambda - \alpha \mid \geqslant r\}$$

are spectral sets for T iff

(1) $\| (T - \alpha I) \| \leqslant r$

(2) $\| (T - I)(T + I)^{-1} \| \leqslant 1$

(3) $\| (T - \alpha I)^{-1} \| \leqslant 1/r$ respectively.

PROOF OF THE THEOREM : $(T - I)$ being unitary,

$$\| (T - I) \| = 1$$

$$\| (T + I)^{-1} (T - I) \| \leqslant \| T - I \| \| (T + I)^{-1} \| \leqslant 1, \text{ by lemma 2.}$$

$\| (T + \mu I)^{-1} \| \leqslant 1/\mu$, $\forall \mu > 0$, also by lemma 2. Thus by lemma 3, sets (I), (II) and (III) are spectral sets for T.

Now as a corollary to Lemma 2 we get the following result.

Theorem 4 — If T is QU and $0 \in W(T)$, then 0 is an eigenvalue for T.

PROOF : Since Re $W(T) \geqslant 0$ and $\overline{W(T)} \subset C_1$, 0 is an extreme point of $W(T)$. Hence by a theorem of MacCluer (1965), 0 is an eigenvalue for T.

If T is a normal operator, any power of T is also normal. This need not be so for a QU operator as the following examples shows. $T = I + V$ where V is the bilateral shift is QU but T^2 is not QU. In fact the following theorem answers the question negatively.

Theorem 5 — If $T \in \beta(H)$ is such that T and T^2 are QU then T is the zero operator.

PROOF : It is easy to see that if T_1, T_2 are QU operators and $T_1 \leftrightarrow T_2$, then $T_1 T_2$ is QU iff $T_2^* \, T_1 + T_1^* \, T_2 = 0$. As T and T^2 are QU we have $T^*T = 0$ i.e. $\| T \|^2 = 0$ or $T = 0$.

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