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ON THE GEOMETRY OF RELATIVISTIC ELECTROMAGNETIC FLUID FLOWS II

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The paper presents the geometrical aspects of the relativistic inductive electromagnetic fluid flows in terms of the kinematical parameters associated with the congruences of space-like and time-like curves. Further, it is shown that the process of gravitational collapse is opposed by the presence of charge and vorticity of the fluid. The magnetic permeability, Joule's heat and electric permitivity are responsible for the generation of entropy.

1. INTRODUCTION

The modern investigations in astronomy and astrophysics have stimulated interest in relativistic matter fields which are more general than the familiar perfect fluid models. Lichnerowicz (1967) initiated fundamental studies of more general hydrodynamical matter field solutions of Einstein's field equations and latter extended his investigation to relativistic magnetohydrodynamics (RMHD). His RMHD field equations are used by Yodzis (1971), Banerji (1974), Date (1976) and Esposito and Glass (1977) to infer the magnetic effect in galactic cosmogony and gravitational collapse. Several aspects of RMHD field equations have been investigated by the author (Prasad and Ojha 1977, Prasad and Sinha 1978, Prasad 1978a) involving the kinematical parameters associated with the congruences of the streamlines, magnetic lines of force and the electric lines of force defined analogous to Ehlers (1961).

Greenberg (1970) has defined the kinematical parameters associated with the congruence of space-like curves and applied his concept to vorticity in relativistic hydrodynamics successfully. Recently the author (Prasad 1978b, c) has used Greenberg's theory in the domain of magnetofluids and established some satisfactory results. But we may ask; how far this theory would be physically valid in the domain of electromagnetic fluids? Thus the purpose of this communication is to examine this question and to abondon the application of kinematical parameters associated with the congruence of space-like curves defined analogous to Ehlers since Greenberg's theory of space-like congruences seems to be more physical than the theory demonstrated in our previous papers (Prasad and Ojha 1977, Prasad and Sinha 1978, Prasad 1978a). In particular, we apply the Greenberg's theory in studying the behaviour of magnetic field tubes and electric field tubes which are the congruences of space-like curves.

2. Kinematical Parameters

The congruence of space-like curves is defined by

$$x^i = x^i(\overset{*}{\eta}{}^\alpha, s^*), \quad (\alpha = 0, 2, 3) \qquad ...(2.1)$$

where x^i $(i = 0, 1, 2, 3)$ is an arbitrary coordinate system for a region of space-time of signature $(+, -, -, -)$, $\overset{*}{\eta}{}^\alpha$ take constant values for a particular space-like curve and s^* is a parameter denoting arc length along this curve. The unit vector tangential to this curve is given by

$$n^i = \frac{dx^i}{ds^*} \qquad ...(2.2)$$

with $\quad n^i n_i = -1.$

The covariant derivative of n^i is decomposed according to Greenberg (1970) as follows:

$$n_{i;j} = \overset{*}{\sigma}_{ij} + \overset{*}{\omega}_{ij} + \overset{*}{\theta}\overset{*}{\gamma}_{ij} - D^* n_i n_j + D n_i u_j$$
$$- (D n_k u^k)\, u_i u_j + (D^* n_k u^k)\, u_i n_j + n_{k;j} u^k u_i \qquad ...(2.3)$$

where $\overset{*}{\sigma}_{ij}, \overset{*}{\omega}_{ij}, \overset{*}{\theta}$ denote the shear, rotation and expansion associated with the congruence of space-like curves respectively. $\overset{*}{\gamma}_{ij}$ is the projection tensor defined by

$$\overset{*}{\gamma}_{ij} = g_{ij} - u_i u_j + n_i n_j \qquad ...(2.4)$$

where u_i is the fluid flow vector whose covariant derivative is decomposed according to Ehlers (1961) as follows :

$$u_{i;j} = \sigma_{ij} + \omega_{ij} + \theta\gamma_{ij} + D u_i u_j \qquad ...(2.5)$$

where $\sigma_{ij}, \omega_{ij}, \theta$ are the shear, rotation and expansion of the congruence of streamlines respectively. D and D^* denote the absolute derivative along the streamline and the magnetic field line defined by the unit space-like vector n^i respectively. We may interpret $\overset{*}{\sigma}_{ij}, \overset{*}{\omega}_{ij}, \overset{*}{\theta}$ as the shear, rotation and expansion of the congruence of magnetic field lines (magnetic field tubes) respectively following the concluding remarks made by Greenberg (1970).

The skew-symmetric tensor dual to $\overset{*}{\omega}_{ij}$ is defined by

$$\underset{\sim}{\overset{*}{\omega}}{}^{ij} = \tfrac{1}{2}\eta^{ijkl} \, . \overset{*}{\omega}_{kl} \qquad ...(2.6)$$

which yields the vector

$$\overset{*}{\omega}{}^i = \tfrac{1}{2}\eta^{ijkl} u_j \overset{*}{\omega}_{kl}. \qquad ...(2.7)$$

It is easy to observe the following properties:

$$\overset{*}{\omega}{}^{i}\overset{*}{\omega}_{i} = -\overset{*}{\omega}{}^{2} \qquad \text{...(2.8)}$$

where

$$2\overset{*}{\omega}{}^{2} = \overset{*}{\omega}_{ij}\overset{*}{\omega}{}^{ij}. \qquad \text{...(2.9)}$$

From (2.8), we may conclude that there exist a space-like vector $\overset{*}{\omega}{}^{i}$ defining the magnetic vorticity vector whose properties will be investigated in detail in next part of this paper which is under preparation.

It is a well-known fact that the fluid flow has the rate of shear and vorticity with components in the 3-space of metric γ_{ij} while the shear and rotation of the space-like congruence reside in 2-space of metric $\overset{*}{\gamma}_{ij}$.

Let us define an alternating tensor on the 3-space quotient to the streamlines by the relation

$$\epsilon_{ijk} = \eta_{ijkl}u^{l} \qquad \text{...(2.10)}$$

with the properties

$$\epsilon_{ijk}\epsilon^{ilm} = 2\gamma^{l}_{[j}\,\gamma^{m}_{k]};\ \epsilon_{ijk}u^{k} = 0 \qquad \text{...(2.11)}$$

where square bracket denotes skew-symmetrization. The alternating tensor on 2-space quotient to the streamlines and magnetic field lines is defined as

$$\overset{*}{\epsilon}_{ij} = \eta_{ijkl}u^{k}n^{l} \qquad \text{...(2.12)}$$

satisfying the following properties:

$$\overset{*}{\epsilon}_{ij}\overset{*}{\epsilon}{}^{ik} = \overset{*}{\gamma}{}^{k}_{j}\ ;\ \overset{*}{\epsilon}_{ij}u^{j} = \overset{*}{\epsilon}_{ij}n^{j} = 0. \qquad \text{...(2.13)}$$

By virtue of (2.12), (2.4) and (2.3), we get

$$\overset{*}{\epsilon}{}^{ij}_{;j} = 2\overset{*}{\omega}{}^{i} - 2u^{i}n_{k}\omega^{k} - \overset{*}{\epsilon}{}^{ik}(Du_{k} - D^{*}n_{k}). \qquad \text{...(2.14)}$$

The above mentioned theory of space-like congruence may also be applied to the study of the behaviour of the congruence of electric field lines (electric field tubes). We put an overhead caret (^) in place of star (*) to distinguish between the kinematical parameters associated with the congruences of electric field lines and of magnetic field lines. With this remark the stage is set to begin our discussion from well known Maxwell's field equations.

3. Maxwell Field Equations

The Maxwell field equations read

$$(u^i B^j - u^j B^i + \eta^{ijkl} u_k e_l)_{;j} = 0, \qquad \ldots(3.1)$$

and

$$(u^i D^j - u^j D^i - \eta^{ijkl} u_k h_l)_{;j} = - J^i, \qquad \ldots(3.2)$$

where B^i is the magnetic induction vector, D^i the electric induction vector, J^i the electric current vector, e^i the electric field vector and h^i the magnetic field vector. The constitutive equations are

$$D^i = \lambda e^i; \; B^i = \mu h^i \qquad \ldots(3.3)$$

where λ is electric permitivity and μ the magnetic permeability.

The electric current vector is decomposed as

$$J^i = \epsilon u^i + \bar{k} e^i \qquad \ldots(3.4)$$

where ϵ is the proper density of electric charge and $\bar{k}$ the electrical conductivity of the fluid.

By virtue of (2.14) and (3.2), we can write

$$\underset{u}{\mathcal{L}} D^i + 3\theta D^i - u^i D^j_{;j} + 2 \mid h \mid \overset{*}{\omega}{}^i - 2u^i h_k \omega^k$$

$$+ \mid h \mid \overset{*}{\epsilon}{}^{ik} \{(\ln \mid h \mid)_{,k} + D^* n_k - D u_k\} = J^i \qquad \ldots(3.5)$$

where $\underset{u}{\mathcal{L}}$ denotes the Lie differentiation with respect to the flow vector u^i and $\mid h \mid > 0$. Similarly (3.1) yields

$$\underset{u}{\mathcal{L}} B^i + 3\theta B^i - u^i B^j_{;j} - 2 \mid e \mid \hat{\omega}^i + 2u^i e_k \omega^k - \mid e \mid \hat{\epsilon}^{ik}$$

$$\times \{(\ln \mid e \mid)_{,k} + \hat{D} a_k - D u_k\} = 0 \qquad \ldots(3.6)$$

where a_k is unit electric field vector and $\mid e \mid > 0$. Contracting (3.6) with u_i, we get

$$D^* \mid B \mid + 2\overset{*}{\theta} \mid B \mid - 2e_k \omega^k = 0 \qquad \ldots(3.7)$$

where

$$\overset{*}{\theta} = \tfrac{1}{2}(n^i_{;i} - n_{i;j} u^i u^j),$$

expansion of the magnetic field tubes.

Using the definition (Greenberg 1970)

$$2\overset{*}{\theta} = D^* \ln A^* \qquad \ldots(3.8)$$

where A^* is the proper area subtended by the magnetic field lines as they pass through the screen in the 2-space quotient to u_i and h_i, in (3.7) we get

$$D^* (\ln \mid B \mid A^*) = 2e_k\omega^k / \mid B \mid , \mid B \mid > 0 \qquad ...(3.9)$$

which states the following theorem.

Theorem 3.1 — The vorticity and electric field are orthogonal iff $\mid B \mid A^*$ is constant along the magnetic field tube.

Equation (3.7) may be regarded as the law of conservation of the magnetic induction along the magnetic field tube. Similarly the law of conservation of electric induction along the electric field tube can be obtained from (3.5) in the form :

$$\hat{D} \mid D \mid + 2\hat{\theta} \mid D \mid + 2h_i\omega^i = - \epsilon, \quad \mid D \mid > 0. \qquad ...(3.10)$$

where $\hat{\theta}$ denotes the expansion of the electric field tubes. The counterpart of (3.9) can be obtained from (3.10) as follows

$$\hat{D} (\ln \mid D \mid \hat{A}) = - (2h_k\omega^k + \epsilon) / \mid D \mid , \quad \mid D \mid > 0. \qquad ...(3.11)$$

Further, (3.9) and (3.11) reveal that the intensity of electric and magnetic fields increases as the area of a contracting medium decreases but the presence of charge and vorticity will oppose this process. Thus we may conclude that the process of gravitational collapse is opposed by the presence of charge and vorticity of the fluid.

The electromagnetic momentum vector (Lichnerowicz 1967) W^i is defined as

$$W^i = \eta^{ijkl} D_j B_k u_l \qquad ...(3.12)$$

which yields

$$W^i_{;i} = (\ln \mid W \mid)_{,i} W^i + 2\mu \mid h \mid D_i \overset{*}{\omega}{}^i - 2\lambda \mid e \mid B_i \hat{\omega}^i$$
$$+ W^i(\hat{D}a_i + D^* n_i) - W^i D u_i \qquad ...(3.13)$$

where

$$W^i W_i = - \mid W \mid ^2.$$

Combining (3.13) and the resulting equations obtained by the contraction of (3.5) with D_i and (3.6) with B_i, we get

$$(\lambda B_i \underset{u}{\mathcal{L}} B^i + \mu D_i \underset{u}{\mathcal{L}} D^i) - 3\theta(\lambda \mid B \mid ^2 + \mu \mid D \mid ^2)$$
$$+ W^i_{;i} - W^i D u_i - (\ln \lambda\mu)_{,i} W^i = - \lambda\mu\bar{k} \mid e \mid ^2 \qquad ...(3.14)$$

which may also be written as

$$\lambda\mu D\{\tfrac{1}{2}(\mu\,|\,h\,|^2 + \lambda\,|\,e\,|^2)\} + \lambda(\tfrac{1}{2}\mu\,|\,h\,|^2)\,D\mu$$
$$+ \mu(\tfrac{1}{2}\lambda\,|\,e\,|^2)\,D\lambda + 2\theta\lambda\mu(\mu\,|\,h\,|^2 + \lambda\,|\,e\,|^2)$$
$$+ \lambda\mu\sigma_{ij}(\lambda e^i e^j + \mu h^i h^j) - W^i_{;i} + W^i Du_i$$
$$+ (\ln \lambda\mu)_{,i} W^i = \lambda\mu\bar{k}\,|\,e\,|^2 \quad \ldots(3.15)$$

and may be called as an electromagnetic energy equation. It is clear from (3.15) that the presence of differential rotation (Esposito and Glass 1977) with electric and magnetic fields causes the electromagnetic energy to be exchanged back and forth for an inductive fluid energy producing Joule's heat.

4. Stress-Energy-Momentum Tensor

A stress-energy-momentum tensor for a thermally conducting, viscous and compressible fluid has a general form (Ellis 1971)

$$\underset{(m)}{T^{ij}} = (\rho + p)\,u^i u^j - pg^{ij} + \nu\sigma^{ij} + q^i u^j + q^j u^i \quad \ldots(4.1)$$

where ρ is the matter energy density of the fluid, p the isotropic pressure, $\nu(\geqslant 0)$ the coefficient of viscosity and q^i the heat energy-flux vector. The matter energy density ρ is connected with the proper energy density i by the relation

$$\rho = \rho_0(1 + i) \quad \ldots(4.2)$$

where ρ_0 is the proper matter density of the fluid.

The relations (Lichnerowicz 1967, Eckart 1940) connecting the thermodynamical variables are

$$TDS = Di + pD(1/\rho_0) \quad \ldots(4.3)$$

$$S^i = \rho_0 S u^i + q^i/T \quad \ldots(4.4)$$

$$q^i = K(T_{,j} - TDu_j)\,\gamma^{ij} \quad \ldots(4.5)$$

and

$$\chi = 1 + i + \frac{p}{\rho_0} \quad \ldots(4.6)$$

where S, T, S^i, K and χ denote the entropy, the rest temperature, the entropy-flux vector, the heat conduction coefficient and the fluid index respectively. The velocity of light is assumed to be unity.

Recently Mason (1976) has given the electromagnetic stress-energy-momentum tensor for an inductive fluid in more simpler form than that originally derived by Quan (1956) as follows :

$$\underset{(em)}{T^{ij}} = \tfrac{1}{2}\{(\lambda\mu + 1)\, u^i u^j - g^{ij}\}(\lambda \mid e \mid^2 + \mu \mid h \mid^2)$$
$$- (\lambda e^i e^j + \mu h^i h^j) - (u^i W^j + u^j W^i) \qquad \ldots(4.7)$$

The total stress-energy-momentum tensor T^{ij} for self-gravitating, thermally conducting, viscous, compressible, inductive and charged fluid is the sum of $\underset{(m)}{T^{ij}}$ and $\underset{(em)}{T^{ij}}$:

$$T^{ij} = (\rho^* + p^*)\, u^i u^j - p^* g^{ij} + \nu\sigma^{ij} - (\lambda e^i e^j + \mu h^i h^j) + Q^i u^j + Q^j u^i \qquad \ldots(4.8)$$

where

$$\rho^* = \rho + \frac{\lambda\mu}{2}(\lambda \mid e \mid^2 + \mu \mid h \mid^2) \qquad \ldots(4.9)$$

$$p^* = p + \tfrac{1}{2}(\lambda \mid e \mid^2 + \mu \mid h \mid^2) \qquad \ldots(4.10)$$

and

$$Q^i = q^i - W^i. \qquad \ldots(4.11)$$

Einstein's field equations read

$$R_{ij} - \tfrac{1}{2} R g_{ij} = - T_{ij}. \qquad \ldots(4.12)$$

The equation of continuity $u_i T^{ij}_{\ ;j} = 0$ yields

$$D\rho^* + 3\theta(\rho^* + p^*) - 2\nu\sigma^2 + (\sigma_{ij} + \theta\gamma_{ij})(\lambda e^i e^j + \mu h^i h^j)$$
$$+ Q^i_{\ ;i} - Q^i D u_i = 0. \qquad \ldots(4.13)$$

Using (4.8) and (4.12) in the relation (Ehlers 1961)

$$\gamma^{ij} R_{jk} u^k = \gamma^i_j (\omega^{jk}_{\ ;k} - \sigma^{jk}_{\ ;k} + 2\theta^{,j}) + (\omega^i_k + \sigma^i_k)\, D u^k \qquad \ldots(4.14)$$

we get

$$Q^i = \gamma^i_j (\omega^{jk}_{\ ;k} - \sigma^{jk}_{\ ;k} + 2\theta^{,j}) + (\omega^i_k + \sigma^i_k)\, D u^k \qquad \ldots(4.15)$$

which shows that the heat energy-flux vector is balanced by the electromagnetic momentum vector when the fluid is free from the expansion, rotation and shear.

In view of (4.13) and (4.15), we observe that

$$D\rho^* = 0 \qquad \ldots(4.16)$$

when the fluid is assumed to be free from expansion, rotation and shear. This shows that the total energy density is constant along the stream line.

By virtue of (3.15) and (4.13), we get

$$D\rho + 3\theta(\rho + p) + \tfrac{1}{2}(1 - \lambda\mu)(\lambda\,|\,e\,|^2 + \mu\,|\,h\,|^2)$$
$$+ (1 - \lambda\mu)\,\sigma_{ij}(\lambda e^i e^j + \mu h^i h^j) - 2\nu\sigma^2$$
$$+ \lambda(\tfrac{1}{2}\mu\,|\,h\,|^2)\,D\mu + \mu(\tfrac{1}{2}\lambda\,|\,e\,|^2)\,D\lambda + \lambda\mu\bar{k}\,|\,e\,|^2$$
$$- (\ln\lambda\mu)_{,i}W^i + q^i_{;i} - q^i Du_i = 0 \qquad \text{...(4.17)}$$

which is the equation of continuity for an inductive, self-gravitating electromagnetic fluid flows. Combining (4.2), (4.3), (4.6) and (4.17), we get the heat transfer equation as follows :

$$\chi(\rho_0 u^i)_{;i} + \rho_0 TDS + \tfrac{1}{2}(1 - \lambda\mu)\,\theta(\lambda\,|\,e\,|^2 + \mu\,|\,h\,|^2)$$
$$+ (1 - \lambda\mu)\,\sigma_{ij}(\lambda e^i e^j + \mu h^i h^j) - 2\nu\sigma^2 + \lambda(\tfrac{1}{2}\mu\,|\,h\,|^2)\,D\mu$$
$$+ \mu(\tfrac{1}{2}\lambda\,|\,e\,|^2)\,D\lambda + \lambda\mu\bar{k}\,|\,e\,|^2 - (\ln\lambda\mu)_{,i}W^i + q^i_{;i}$$
$$- q^i Du_i = 0. \qquad \text{...(4.18)}$$

Assuming that the matter density is conserved, i.e., $(\rho_0 u^i)_{;i} = 0$ and using (4.4), (4.5) in (4.18), we have

$$S^i_{;i} = \frac{1}{T}\,[2\nu\sigma^2 + |\,q\,|^2/KT + \tfrac{1}{2}(\lambda\mu - 1)\,\theta(\lambda\,|\,e\,|^2 + \mu\,|\,h\,|^2)$$
$$+ (\lambda\mu - 1)\,\sigma_{ij}(\lambda e^i e^j + \mu h^i h^j) + (\ln\lambda\mu)_{,i}W^i$$
$$- \lambda\mu\bar{k}\,|\,e\,|^2 - \lambda(\tfrac{1}{2}\mu\,|\,h\,|^2)\,D\mu - \mu(\tfrac{1}{2}\lambda\,|\,e\,|^2)\,D\lambda] \qquad \text{...(4.19)}$$

which reveals that the generation of entropy depends upon the Joule's heat, magnetic permeability and electric permitivity of the fluid besides the expansion, shear and electromagnetic energy density of the fluid and thermal heat. Equation (4.19) may be regarded as a generalization of the result obtained by Date (1976). The entropy generation for a self-gravitating, thermally conducting, viscous, inductive, compressible and charged fluid is positive only if following inequality

$$2\nu\sigma^2 + |\,q\,|^2/KT + \tfrac{1}{2}(\lambda\mu - 1)\,\theta(\lambda\,|\,e\,|^2 + \mu\,|\,h\,|^2)$$
$$+ (\lambda\mu - 1)\,\sigma_{ij}(\lambda e^i e^j + \mu h^i h^j) + (\ln\lambda\mu)_{,i}W^i \geqslant \lambda\mu\bar{k}\,|\,e\,|^2$$
$$+ \lambda(\tfrac{1}{2}\mu\,|\,h\,|^2)\,D\mu + \mu(\tfrac{1}{2}\lambda\,|\,e\,|^2)\,D\lambda$$

holds.

References

Banerji, S. (1974). The perfectly conducting medium in general relativity. *Nuovo Cim.*, **23B**, 345.
Date, T. H. (1976). On the relativistic magnetofluids. *Ann. Inst. Henri Poincare*, **24**, 417.

Eckart, C. (1940). The thermodynamics of irreversible process II: Relativistic theory of simple fluids. *Phys. Rev.*, **58**, 919.

Ehlers, J. (1961). *Akad. Wiss, Lit. Mainz. Abh. Math. Nat. Kl.*, **11**, 1.

Ellis, G. F. R. (1971). General Relativity and Cosmology. Academic Press, New York.

Esposito, F. P., and Glass, E. N. (1977). General relativistic magnetofluids. *J. Math. Phys.*, **18**, 705.

Greenberg, P. J. (1970). The general theory of space-like congruences with an application to vorticity in relativistic hydrodynamics. *J. Math. Analysis Applic.*, **30**, 128.

Lichnerowicz, A. (1967). Relativistic Hydrodynamics and Magnetohydrodynamics. Benjamin, N. Y., p. 24.

Mason, D. P. (1976). Characteristic hypersurfaces of a charged inductive fluid in general relativity. *Tensor, N. S.*, **30**, 311.

Prasad, G. (1978a). On the geometry of relativistic electromagnetic fluid flows with geometrical symmetries. *Indian J. pure appl. Math.*, **9**, 457.

———— (1978b). Relativistic magnetofluids and symmetry mappings I. *Indian J. pure appl. Math.*, **9**, 682.

———— (1978c). Relativistic magnetofluids and symmetry mappings II. *Indian J. pure appl. Math.*, **9**, 692.

Prasad, G., and Ojha, S. N. (1977). Tetrad formalism of Maxwell equations with effect of gravitation II. *Indian J. pure appl. Math.*, **8**, 741.

Prasad, G., and Sinha, B. B. (1978). On the geometry of relativistic electromagnetic fluid flows I. *Indian J. pure appl. Math.*, **9**, 447.

Quan, P. M. (1957). Inductions electromagnetiques en relativite generale. *Arch. ration Mech. Analysis*, **1**, 54.

Yodzis, P. (1971). Some general relations in relativistic magnetohydrodynamics. *Phys. Rev.*, D **3**, 2941.