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GENERALIZED FOURIER SERIES AND NON-LINEAR OSCILLATOR

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1. INTRODUCTION

It is well known that the straight forward expansion method is not valid in case of non-linear oscillators. A number of techniques have been developed in the past to overcome this difficulty (for example see Nayfeh 1973, Giacaglia 1972).

In this note we apply a generalized Fourier series expansion to obtain solution of any order for the Duffing non-linear oscillator.

2. METHOD OF SOLUTION

Let us consider the Duffing non-linear oscillator which has the following equation of motion

$$\frac{d^2x}{dt^2} + \omega_0^2 x = -\epsilon x^3 \qquad \text{...(1)}$$

with $x(0) = a$ and $\dot{x}(0) = 0$

We seek a solution of (1) in the form

$$x(t) = \sum_{n=1,3,5,\text{ etc.}} a_n \epsilon^{(n-1)/2} \cos n\omega t \qquad \text{...(2)}$$

where

$$a_n = \sum_{m=0}^{n-1} a_{mn} \epsilon^m \qquad \text{...(3)}$$

and

$$\omega = \sum_{m=0}^{\infty} \omega_m \epsilon^m. \qquad \text{...(3a)}$$

a_n may depend on t but we consider ω to be independent of t to prevent secular term appearing on the l. h. s. of (1) when the expression (2) would be used. Also the first term in the expansion (3a) is identical with ω_0 of eqn. (1).

The reason for taking only odd integer values in the expansion (2) will be clear from the fact that r. h. s. of (1) contains an odd power of x and also

$$\cos^3 z = \tfrac{1}{4}\,[3 \cos z + \cos 3z] \qquad ...(4)$$

Of course, one can take a more general Fourier series instead of (2) and would arrive at the same results as with (2), though the calculations will be slightly more lengthy.

Putting (2) in (1), we get

$$\sum_n (\ddot{a}_n + (\omega_0^2 - n^2\omega^2)) \cos n\omega t\, \epsilon^{(n-1)/2} - 2\sum_n n\omega \dot{a}_n \sin n\omega t\, \epsilon^{(n-1)/2}$$

$$= -\epsilon \sum_l \sum_m \sum_n \epsilon^{(l+m+n-3)/2} a_l a_m a_n \cos n\omega t \cos m\omega t \cos l\omega t \qquad ...(5)$$

$$= -\frac{\epsilon}{4} \sum_l \sum_m \sum_n a_l a_m a_n\, \epsilon^{(l+m+n-3)/2}$$

$$\times [\cos (m+n+l)\,\omega t + \cos (m+n-l)\,\omega t + \cos (m-n+l)\,\omega t + \cos (m-n-l)\,\omega t] \qquad ...(6)$$

where a dot denotes derivative with respect to t.

Since right-hand side of (6) does not contain any sine term

$$\dot{a}_n = 0 \quad \text{for all} \quad n. \qquad ...(7)$$

Hence a_{mn} are independent of t also,

$$\ddot{a}_n = 0. \qquad ...(8)$$

Therefore, multiplying both the sides of (6) by $\cos k\omega t$ ($k = 1, 3, 5...$) and integrating with respect to t from 0 to π, we have

$$(\omega_0^2 - k^2\omega^2)\, a_k\, \epsilon^{(k-1)/2} = -\frac{\epsilon}{4} \Big[\sum_l \sum_m \sum_n \epsilon^{(l+m+n-3)/2} a_l a_m a_n$$

$$\times (\delta_{k,m+n+l} + \delta_{k,|m+n-l|} + \delta_{k,|m-n+l|} + \delta_{k,|m-n-l|}) \quad (k = 1, 3, 5, ...) \qquad ...(9)$$

where δ_{ij} denotes the Kronecker delta defined by

$$\delta_{ij} = \begin{matrix} 1 \\ 0 \end{matrix} \Big\} \begin{matrix} i = j \\ i \neq j. \end{matrix} \qquad ...(10)$$

Putting $k = 1, 3, 5$ respectively, we get from (9)

$$(\omega_0^2 - \omega^2)\, a_1 = -\frac{\epsilon}{4}\, [3a_1^3 + 3a_1^2 a_3 \epsilon + 6\epsilon^2 a_3^2 a_1 + O(\epsilon^3)] \qquad ...(11)$$

$$\epsilon(\omega_0^2 - 9\omega^2)\, a_3 = -\frac{\epsilon}{4}\, [a_1^3 + 6a_1^2\, a_3\, \epsilon + 3a_1^2\, a_5\, \epsilon^2 + O(\epsilon^3)] \qquad ...(12)$$

$$\epsilon^2(\omega_0^2 - 25\omega^2)\, a_5 = -\frac{\epsilon}{4}\, [3a_1^2\, a_3\, \epsilon + 6\,\epsilon a_1^2\, a_5 + 3\,\epsilon^2 a_3^2\, a_1 + O(\epsilon^3)] \qquad ...(13)$$

Taking $a_1 = a_{01} = a$ and equating terms of the first order we get from (11) and (12)

$$-\,2a\omega_0\omega_1 = -\,\tfrac{3}{4}\, a^3 \qquad ...(14)$$

and

$$-\,8\omega_0^2\, a_{03} = -\,\tfrac{1}{4}a^3. \qquad ...(15)$$

Solving them we get

$$\omega_1 = 3a^2/8\omega_0 \qquad ...(16)$$

$$a_{03} = a^3/32\omega_0^2. \qquad ...(17)$$

Hence, to the first order,

$$x = a \cos \omega t + \frac{\epsilon a^3 \cos 3\omega t}{32\omega_0^2} + O(\epsilon^2) \qquad ...(18)$$

where

$$\omega = \omega_0 \left(1 + \frac{3a^2\epsilon}{8\omega_0^2}\right) \qquad ...(19)$$

a result obtained by various other methods, for example, the method of strained coordinates due to Lindstedt (1882) and Poincare (1967), the method of averaging by Krylov and Bogoliubov (1947) and Bogoliubov and Mitropolski (1961) and the method of coherent state constructed out of quantum oscillator proposed by Bhaumik and Dutta Roy (1975).

If we equate the terms of 2nd order in ϵ, then we get from (11), (12) and (13), three equations involving ω_3 and a_{13}, a_{05}. Solving them we get,

$$\omega_2 = -\,15a^4/256\omega_0^3 \qquad ...(20)$$

$$a_{13} = -\,21a^5/1024\omega_0^4 \qquad ...(21)$$

and

$$a_{05} = a^5/1024\omega_0^4. \qquad ...(22)$$

Hence, to the second order

$$x = a \cos \omega t + \frac{\epsilon a^3}{32\omega_0^2} \cos 3\omega t - \frac{\epsilon^2 a^5}{1024\omega_0^4} (21 \cos 3\omega t - \cos 5\omega t) + O(\epsilon^3) \qquad ...(23)$$

where

$$\omega = \omega_0\left(1 + \frac{3\,\epsilon a^2}{8\omega_0^2} - \frac{15}{256}\,\frac{\epsilon^2 a^4}{\omega_0^4}\right). \qquad ...(24)$$

3. Higher Order Terms

The solution given by (23) and (24) is identical with that obtained by Struble (1962). But to get third order terms, Struble's method becomes very much involved. If one wishes, one can obtain terms up to any order very easily by our methods. One has to put simply $k = 7, 9, \ldots$ etc., and solve the subsequent equations for a_{mn}.

For example, if we put $k = 7$ in (9) we get

$$\epsilon^3(\omega_0^2 - 49\omega^2)\,a_7 = -\frac{\epsilon}{4}\,[3\,\epsilon^2 a_1^2\, a_5 + O(\epsilon^3)] + 3\,\epsilon^2 a_3^2\, a_1 \qquad ...(25)$$

Now equating terms of order ϵ^3 from both the sides of eqns. (11) - (13) we have

$$-2\omega_1\omega_2 - 2\omega_0\omega_3 = -\tfrac{1}{4}\,[3a^2 a_{13} + 6a_{03}^2\, a] \qquad ...(26)$$

$$-9\omega_1^2\, a_{03} - 8\omega_0^2\, a_{23} - 18\omega_0\omega_1 a_{13} - 18\omega_0\omega_2 a_{03} = -\tfrac{1}{4}\,[6a^2 a_{13} + 3a^2 a_{05}] \qquad ...(27)$$

$$-24\omega_0^2\, a_{15} - 50\omega_0\omega_1 a_{05} = -\tfrac{1}{4}\,[3a^2 a_{13} + 6a^2 a_{05} + 3a_{03}^2\, a] \qquad ...(28)$$

$$-48\omega_0^2\, a_{07} = -\tfrac{3}{4}\,(a^2 a_{05} + a_{03}^2\, a). \qquad ...(29)$$

Solving eqns. (26) - (28) we get

$$\omega_3 = 123a^6/8.1024\omega_0^5 \qquad ...(29a)$$

$$a_{23} = 834a^7/64.1024\omega_0^6 \qquad ...(30)$$

$$a_{07} = 2a^7/64.1024\omega_0^6 \qquad ...(31)$$

$$a_{15} = -\,86/64.1024\omega_0^6. \qquad ...(32)$$

Hence, to the third order

$$x = a\cos\omega t + \frac{\epsilon a^3}{32\omega_0^2}\cos 3\omega t - \frac{\epsilon^2 a^5}{1024\omega_0^4}\,(21\cos 3\omega t - \cos 5\omega t)$$

$$+ \frac{\epsilon^3 a^7}{2\times 32^3\omega_0^6}\,(834\cos 3\omega t - 86\cos 5\omega t + 2\cos 7\omega t) + O(\epsilon^4) \qquad ...(33)$$

where

$$\omega = \omega_0\left[1 + \frac{3\epsilon a^2}{8\omega_0^2} - \frac{15\epsilon^2 a^4}{256\omega_0^4} + \frac{123\epsilon^3 a^6}{8192\omega_0^6} + O(\epsilon^4)\right]. \qquad ...(34)$$

4. Validity and Rate of Convergence of the Fourier Series

To investigate the convergence of the series (2) the relevant parameter which should be examined is not just ϵ but the quantity $\epsilon a^2/\omega_0^2$ as can be seen from the following. If we put $\tau = \omega_0 t$, then (1) can be written as (putting $x = au(t)$, $u(0) = 1$)

$$\frac{d^2u}{dt^2} = - u(1 + \lambda u^2) \qquad ...(35)$$

where $\lambda = \dfrac{\epsilon a^2}{\omega_0^2}$.

For $-1 < \lambda < \infty$, the solution of (35) is bounded and oscillatory. Let us look at the convergence of (2) as a solution of (1).

Let us write

$$u(t) = \sum_{n=0}^{\infty} \lambda^n u_n(t). \qquad ...(36)$$

Now it is clear from (2) that the coefficients of all powers of λ in (36) are bounded ($\because$ $|\cos z| \leqslant 1$) for all values of t. Hence $u_n(t)$ is no more singular than $u_{n-1}(t)$ and the expansion is uniformly valid as a consequence.

To find the rate of convergence we calculate the quantities $\left|\dfrac{u_n(t)}{u_0(t)}\right|$. From (33) it is found that

$$u_0(t) = a \cos \omega t \qquad ...(37)$$

$$u_1(t) = \frac{1}{32} \cos 3\omega t \qquad ...(38)$$

and so on.

Using $$\left|\frac{\cos (2n + 1) z}{\cos z}\right| \leqslant (2n + 1) \qquad ...(39)$$

we have

$$\left|\frac{u_1}{u_0}\right| \leqslant \frac{3}{32} \simeq 0.094 \qquad ...(40)$$

and similarly

$$\left|\frac{u_2}{u_0}\right| \leqslant \frac{17}{256} \simeq 0.0663$$

$$\left|\frac{u_3}{u_0}\right| \leqslant \frac{3921}{64.1024} \simeq 0.0598$$

Though the general ratio can be estimated numerically for any n, from the recurrence relation (9) one can infer roughly that the subsequent ratio will be smaller than the preceding one from the following fact. From (9) we see that a_k will be proportional to $1/(\omega_0^2 - k^2\omega^2)$ whose magnitude decreases with k.

Hence at least for $|\lambda| < 1$ the Fourier series (2) as a solution of (1) will be convergent. To test the convergence for $\lambda > 1$ one should expand about a point $\lambda = \lambda_0$, where λ_0 is a positive number and investigate the convergence of the solution of (1) by introducing a frequency ν different from ω_0 (For a quantitative discussions see Eminhizer *et al.* 1976).

(*Note* : A difference in sign in the definition of ϵ and also our series is basically different from theirs in the sense that we expand each a_n in terms of ϵ again).

5. Discussion and Conclusion

It is shown that a generalised fourier series provides an easy way to solve the Duffing non-linear oscillator up to any order. Secular terms can be avoided by choosing ω, independent of τ, from the beginning which proved subsequently to be consistent with the equations for expansion parameters. Also, as shown in section 4, the Fourier series (2) as a solution of (1) is convergent at least for $|\lambda| < 1$ which means that $|\epsilon| < \omega_0^2/a^2$. Though the method is applied in this case to Duffing non-linear oscillator, it can also be applied to, for example, the Van Der Pol oscillator

$$\frac{d^2u}{dt^2} + u = \epsilon(1 - u^2)\frac{du}{dt}$$

in which case the calculations will be more lengthy as one has to consider sine terms in the generalized Fourier series expansion given by (2) [the sine terms appear due to the term $\dot{u}$ on the right-hand side of (34)]. However, the power of u still being odd one considers only odd cosine and sine terms and obtains two sets of equations for the expansion coefficients and thus solutions can be obtained in a straight forward manner.

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