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MAGNETOHYDRODYNAMIC UNSTEADY FLOW OF A MAXWELL FLUID PAST A FLAT PLATE

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In this paper the study of the equations describing the flow pattern set up in a linear eletrically conducting viscoelastic fluid past an infinite flat plate in the presence of a transverse magnetic field when the plate is moving parallel to itself with an arbitrary time dependent velocity has been made. The pressure is assumed to be uniform with initial velocity distribution in an exponential form. Operational methods are used to obtain the exact solutions for the velocity profiles. The effects of relaxation parameter of the fluid and magnetic field have been studied. Several particular cases can easily be deduced. The following two cases are discussed :

(i) When the plate is moving in its own plane harmonically with time; and

(ii) when the velocity of the plate is decaying exponentially with time.

INTRODUCTION

Stokes (1850) studied the problem of the motion engendered in a fluid by the action of solid body moving relative to it theoretically for the first time. Na and Sidhom (1967) obtained some interesting results for flow of Maxwell type fluids near an oscillating plate. The same problem was considered by Prakash (1971) for viscous fluids under generalized boundary conditions with specific initial condition. Roy (1978) extended the problem of Prakash (1971) for viscoelastic fluid of Maxwell type. Such problems are still a point of attention of workers in the field and the present paper is the initiative of this attraction. Magnetohydrodynamics effects are also visualized for the flow in a linear viscoelastic fluid past an infinite flat plate.

In this paper we study the flow of an electrically conducting viscoelastic fluid of Maxwell type in the presence of a uniform transverse magnetic field. It is assumed that the fluid properties are uniform in time and space, the flow laminar, surface smooth and the 'no slip' condition exists at the surface i.e. the fluid in the immediate vicinity of the surface moves with the surface. The plate is taken electrically non-conducting and moving parallel to itself with time dependent velocity under uniform pressure with the initial distribution of velocity in an exponential form. Exact solutions for the velocity profiles have been obtained in the general form by using operational methods.

Basic Equations and Statement of Problem

If P is the static pressure, g^{ij} the associated metric tensor and P^{ij} a tensor usually related to the rate of strain, then the stress tensor for a linear, isotropic, viscoelastic fluid is given by (Fredrickson 1964)

$$S^{ij} = - Pg^{ij} + P^{ij} \qquad \text{...(1)}$$

and if P' and Q' are two operators defined by

$$P' = 1 + \lambda_0 \frac{d}{dt} + \lambda_1 \frac{d^2}{dt^2} + \ldots + \lambda_n \frac{d^{n+1}}{dt^{n+1}} \qquad \text{...(2)}$$

$$Q' = \mu \left(1 + S_0 \frac{d}{dt} + S_1 \frac{d^2}{dt^2} + \ldots + S_n \frac{d^{n+1}}{dt^{n+1}}\right) \qquad \text{...(3)}$$

then e^{ij}, a tensor related to the equation of state is given by

$$P'P^{ij} = 2Q'e^{ij} \qquad \text{...(4)}$$

where $S_0, S_1, \ldots S_n; \lambda_0, \lambda_1, \ldots \lambda_n$ are constants. μ is the viscosity of the material at zero rate of shear and d/dt is the convected derivative.

A special case of eqn. (4) which obey the following equation of state :

$$\left(1 + \lambda_0 \frac{d}{dt}\right) P^{ij} = 2\mu e^{ij} \qquad \text{...(5)}$$

is known as Maxwell fluid. The physical significance of λ_0 (the relaxation time) is that if the motion stops suddenly, the shear stress will decay as exp $(-t/\lambda_0)$.

Consider the unsteady flow of an electrically conducting Maxwell fluid over a infinite flat plate at the $y = 0$ plane such that the x-axis is along the plate and parallel to the flow. A uniform and constant magnetic field H_0 is imposed along y-axis. The plate is taken electrically non-conducting and the magnetic permeability μ_e is constant throughout the field. As the plate is infinite in length, all variables in this problem are function of y and t only.

The momentum equation for unsteady electrically conducting Maxwell fluid in the presence of transverse magnetic field, leaving the induced magnetic field can be written as

$$\frac{\partial u}{\partial t} = \frac{1}{\rho} \frac{\partial}{\partial y} (P_{xy}) - \frac{\sigma B_0^2}{\rho} u \qquad \text{...(6)}$$

where $B_0 = \mu_e H_0$, u is the component of the fluid velocity in the direction of x-axis, ρ the density of the fluid and P_{xy} the physical component of P^{ij} defined by

$$\left(1 + \lambda_0 \frac{d}{dt}\right) P_{xy} = u \frac{\partial u}{\partial y} \cdot \qquad \text{...(7)}$$

The initial and boundary conditions of the problem under consideration are

$$u = A \exp(-NY) \quad \text{at} \quad t = 0 \quad \text{for} \quad y \geqslant 0 \qquad \ldots(8)$$

$$\left.\begin{array}{llll} u = f(t) & \text{at} \ \ y = 0 & \text{for} & t > 0 \\ u \to 0 & \text{as} \ \ y \to \infty & \text{for} & t \geqslant 0 \end{array}\right\} \qquad \ldots(9)$$

where A and N are non-negative constants and $f(t)$ is a bounded continuous or piece-wise continuous arbitrary function of time.

Solution of Equations

The problem is solved by using Laplace transform which is defined as follows :

$$\bar{u}(y, p) = \int_0^\infty \exp(-pt)\, u(y, t)\, dt, \ (p > 0) \qquad \ldots(10)$$

and its inversion as

$$u(y, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(pt)\, \bar{u}(y, p)\, dp \qquad \ldots(11)$$

Now to solve the equations under given boundary conditions, multiply (6) and (7) by $\exp(-pt)$ and then integrate from 0 to ∞ by using the initial conditions. Thus we have

$$\left(p + \frac{\sigma B_0^2}{\rho}\right) \bar{u} - Ae^{-NY} = \frac{1}{\rho} \frac{d}{dy} \bar{P}_{xy} \qquad \ldots(12)$$

and

$$(1 + \lambda_0 p)\, \bar{P}_{xy} = \mu \frac{d\bar{u}}{dy} \qquad \ldots(13)$$

where

$$\bar{u} = \int_0^\infty \exp(-pt)\, u(y, t)\, dt$$

and

$$\bar{P}_{xy} = \int_0^\infty \exp(-pt)\, P_{xy}\, dt.$$

It is assumed that P_{xy} vanishes initially. Eliminating P_{xy} from (12) and (13) we have

$$\nu \frac{d^2\bar{u}}{dy^2} - (1 + \lambda_0 p)(p + m)\, \bar{u} = -A(1 + \lambda_0 p) \exp(-NY) \qquad \ldots(14)$$

where $\nu = \mu/\rho$ is the kinematical coefficient of viscosity and $m = \sigma B_0^2/\rho$.

The boundary conditions are

$$\left.\begin{array}{ll} \bar{u} = \bar{f}(p) & \text{at} \quad y = 0 \\ \bar{u} \to 0 & \text{as} \quad y \to \infty. \end{array}\right\} \qquad \ldots(15)$$

Solving (14) subject to boundary conditions (15) we have

$$\bar{u} = \bar{f}(p) \exp(-SY) + \frac{1 + \lambda_0 p}{\nu(S^2 - N^2)} \{\exp(-NY) - \exp(-SY)\} \qquad \ldots(16)$$

where

$$S^2 = (1 + \lambda_0 p)(m + p)/\nu.$$

Finally Laplace inversion of (16) leads (Carslaw and Jaeger 1949)

$$u(y, t) = \frac{A \exp(-NY)}{\lambda_0(a_1 - a_2)} [(1 + \lambda_0 a_1) \exp(a_1 t) - (1 + \lambda_0 a_2) \exp(a_2 t)], \quad 0 \leqslant t \leqslant y/k \qquad \ldots(17)$$

and

$$u(y, t) = f\left(t - \frac{y}{k}\right) \exp\left(\frac{-c_1 y}{k}\right) + \frac{c_2 y}{k} \int_{y/k}^{t} f(t - r) \exp(-c_1 r) \times \frac{I_1\left[c_2\left(r^2 - \frac{y^2}{k^2}\right)^{1/2}\right]}{\left(r^2 - \frac{y^2}{k^2}\right)^{1/2}} dr + \psi(y, t) + g(y, t) \qquad \ldots(18)$$

where

$$\psi(y, t) = \frac{A}{\lambda_0(a_1 - a_2)} \left[(1 + \lambda_0 a_2) \exp\left(a_2 t - \frac{a_2 y}{k} - \frac{c_1 y}{k}\right) - (1 + \lambda_0 a_1) \exp\left(a_1 t - \frac{a_1 y}{k} - \frac{c_1 y}{k}\right) + \frac{c_2 y}{k} \int_{y/k}^{t} \{(1 + \lambda_0 a_2) \exp(a_2 t - a_2 r - c_1 r) - (1 + \lambda_0 a_1) \exp(a_1 t - a_1 r - c_1 r)\} \frac{I_1\left[c_2\left(r^2 - \frac{y^2}{k^2}\right)^{1/2}\right]}{\left(r^2 - \frac{y^2}{k^2}\right)^{1/2}} dr\right] \qquad \ldots(19)$$

$$g(y, t) = \frac{A \exp(-NY)}{\lambda_0(a_1 - a_2)} [(1 + \lambda_0 a_1) \exp(a_1 t) - (1 + \lambda_0 a_2) \exp(a_2 t)] \quad ...(20)$$

$$K = \sqrt{\frac{\nu}{\lambda_0}}, \; c_1 = \frac{1 + m\lambda_0}{2\lambda_0}, \; c_2 = \frac{1 - m\lambda_0}{2\lambda_0}$$

and I_1 denotes the modified Bessel function of first kind and of order unity. a_1 and a_2 are the roots of the equation $\lambda_0 s^2 + (1 + \lambda_0 m) S + (m - \nu N^2) = 0$ and are given by

$$a_1, a_2 = \frac{-(1 + \lambda_0 m) \pm [(1 + \lambda_0 m)^2 - 4\lambda_0(m - \nu N^2)]^{1/2}}{2\lambda_0}.$$

Now we discuss the character of the motion exhibited by eqns. (17) and (18). We observe that the first term in eqn. (18) corresponds to a propogation of the input disturbance in the direction of increasing y with velocity k. This leads that the disturbance produced by an oscillating plane in a viscoelastic fluid has a finite velocity of propogation whose amplitude falls of exponentially with the distance from the plate, having dropped be e^{-1} over a distance $y = k/c_1$. It is also clear that velocity falls as m (magnetic field) increases. The second term in (18) represents, the cumulative effect of the driving function $f(r)$ during the interval $r = 0$ to $r = t - y/k$. The eqn. (17) and the last term of (18) i.e. $g(y, t)$ is the outcome of initial distribution.

It is also seen that a_1, a_2, $-c_1$, c_2 decreases as m increases and hence velocity decreases as m increases which implies that the magnetic field is not favourable to the flow. It is also clear that initial distribution modifies the velocity profiles. Hence, the velocity is dependent on the motion of the plate, magnetic field and the initial distribution of the velocity.

The above results are in complete agreement with that obtained by Roy (1978) if we put $m = 0$. Corresponding results for the ordinary viscous case may be obtained by putting $\lambda_0 = 0$.

The results for a plate moving suddenly in its own plane with a constant velocity U_0 in the electrically conducting fluid can be obtained from (17) and (18) by substituting $f(t) = U_0$ and $A = 0$.

Particular Cases

Case I : When the Plate is Moving in Its own Plane Harmonically with Time

Let $f(t) = U_0 \cos wt$ ...(21)

where $U_0 > 0$, $w > 0$ are constants.

Substituting $f(t)$ in (17) and (18) we have

$$u(y, t) = g(y, t) \quad \text{for} \quad 0 \leqslant t \leqslant \frac{y}{k}$$

and

$$u(y, t) = U_0 \cos\left\{w\left(t - \frac{y}{k}\right)\right\} \exp\left(-\frac{c_1 y}{k}\right)$$

$$+ \frac{c_2 y U_0}{k} \int_{y/k}^{t} \cos\{w(t-r)\} \exp(-c_1 r) \frac{I_1\left[c_2\left(r^2 - \frac{y^2}{k^2}\right)^{1/2}\right]}{\left(r^2 - \frac{y^2}{k^2}\right)^{1/2}} dr$$

$$+ \psi(y, t) + g(y, t), \quad \text{for} \quad t > \frac{y}{k} \qquad \text{...(22)}$$

where $\psi(y, t)$ and $g(y, t)$ are given by eqns. (19) and (20) respectively.

The first term in eqn. (22) shows that in addition to the variation of amplitude with y, there is also a variation in the phase of velocity wave. Other terms may be discussed in the same way as done earlier.

Case II : *When the Velocity of the Plate is Decaying Exponentially with Time*

Let $f(t) = U_0 e^{-wt}$

Substituting the values of $f(t)$ in (17) and (18) we have

$$u(y, t) = g(y, t) \quad \text{for} \quad 0 \leqslant t \leqslant \frac{y}{k}$$

and

$$u(y, t) = U_0 \exp\left(-wt + w\frac{y}{k} - \frac{c_1 y}{k}\right)$$

$$+ \frac{c_2 U_0 y}{k} \int_{y/k}^{t} \exp(-wt + wr - c_1 r) \frac{I_1\left[c_2\left(r^2 - \frac{y^2}{k^2}\right)^{1/2}\right]}{\left(r^2 - \frac{y^2}{k^2}\right)^{1/2}} dr$$

$$+ \psi(y, t) + g(y, t), \quad \text{for} \quad t > \frac{y}{k} \qquad \text{...(23)}$$

where $\psi(y, t)$ and $g(y, t)$ are the same as given by eqns. (19) and (20) respectively.

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