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ON THE DEGREE OF APPROXIMATION TO A FUNCTION BY THE CÈSARO MEANS OF ITS FOURIER-LAGUERRE EXPANSIONS

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In this paper the degree of approximation of the generating function $f(x)$, by the Cèsaro mean of order k of its Fourier Laguerre series at the frontier point $x = 0$ is obtained.

§1. The Fourier-Laguerre expansion associated with the function $f(x) \in L(0, \infty)$ is given by

$$f(x) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \qquad \text{...(1.1)}$$

where

$$a_n = \left[(\alpha + 1)^{1/2} \binom{n+\alpha}{n}\right]^{-1} \int_0^{\infty} e^{-y} y^{\alpha} L_n^{(\alpha)}(y) f(y)\, dy. \qquad \text{...(1.2)}$$

$L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of order $\alpha > -1$. We write

$$\phi(u) = \frac{e^{-y} y^{\alpha} [f(y) - f(0)]}{(\alpha + 1)^{1/2}}$$

$$\Phi_p(y) = \frac{1}{p^{1/2}} \int_0^{y} (y - u)^{p-1} \phi(u)\, du, \; p > 0$$

$$\Phi_0(y) = \phi(y)$$

$$\phi_p(y) = (\alpha + 1)^{1/2} y^{-p} \Phi_p(y), \; p \geqslant 0$$

and

$$\Phi_p(y) = \frac{d}{dy} \Phi_{p+1}(y), \; -1 < p < 0.$$

§2. Gupta (1971) has extended to Laguerre series a classical result of Sunouchi (1951) on Fourier-Trigonometric series to estimate the degree of approximation. Generalizing the result of Gupta (1971), Singh (1975) proved the following:

Theorem — If $p \geqslant 0$, $\epsilon \geqslant 0$ and

$$F(t) \equiv \int_0^t | \phi_p(u) | \, du = o\left[t^{\alpha+1}\left(\log \frac{1}{t}\right)^{\epsilon}\right] \qquad \ldots(2.1)$$

as $t \to 0$, then

$$\sigma_n^k (0) = f(0) + o[\log n]^{\epsilon}$$

as $n \to \infty$, provided that $\alpha > -1$ and $k > \alpha + p + \frac{1}{2}$, and

$$\int_1^{\infty} e^{u/2} u^{(-3k-1)/3} \, | \phi(u) | \, du < \infty. \qquad \ldots(2.2)$$

§3. The object of the present paper is to establish the following:

Theorem — If $p \geqslant 0$, $\epsilon \geqslant 0$ and

$$\int_t^{\delta} \frac{| \phi_p(y) |}{y^{(2\alpha+2k+3-2p)/4}} \, dy = o\left[\left(\log \frac{1}{t}\right)^{\epsilon}\right] \qquad \ldots(3.1)$$

as $t \to 0$, and

$$\int_1^{\infty} e^{y/2} y^{-(6\alpha+6k+6p+7)/12} \, | \phi(y) | \, dy < \infty \qquad \ldots(3.2)$$

then

$$\sigma_n^k (0) - f(0) = o\,[n^{(2\alpha+2p+1-2k)/4} \, (\log n)^{\epsilon}]$$

$n \to \infty$, provided that, $\alpha > -1$ and $k \geqslant \alpha + p + \frac{1}{2}$.

§4. We require the following lemma in the proof.

Lemma — If (3.1) holds, then

$$\int_0^t | \phi_p(y) | \, dy = o\left[t^{(2\alpha+2k-2p+3)/4}\left(\log \frac{1}{t}\right)^{\epsilon}\right]. \qquad \ldots(4.1)$$

PROOF: Let

$$F(t) = \int_t^{\delta} \frac{| \phi_p(y) |}{y^{(2\alpha+2k-2p+3)/4}} \, dy.$$

We set

$$\int_0^t | \phi_p(y) | \, dy = \int_0^t y^{(2\alpha+2k-2p+3)/4} \, \frac{| \phi_p(y) |}{y^{(2\alpha+2k-2p+3)/4}}.$$

Now on integration by parts and using (3.1) we estimate the integral equal to

$$o\left[t^{(2\alpha+2k-2p+3)/4}\left(\log\frac{1}{t}\right)^{\epsilon}\right]$$

§5. *Proof of the Theorem* — The case $p = 0$ is the same as that of Gupta (1971), here we discuss the theorem for $p > 0$. We have (Szegö 1958, p. 270)

$$\sigma_n^k(0) = (A_n^k)^{-1}\int_0^\infty L_n^{(\alpha+k+1)}(y)\,\phi(y)\,dy$$

$$= \int_0^\eta + \int_\eta^1 + \int_1^\infty = I_1 + I_2 + I_3 \qquad \ldots(5.1)$$

say, where η is fixed but small and suitably chosen real number.

Let us consider I_1 integrating by parts m times, we get

$$I_1 = (A_n^k)^{-1}\left[\left\{\sum_{h=1}^{m}(-1)^{h-1}\,\Phi_h(y)\left(\frac{d}{dy}\right)^{h-1}L_n^{(\alpha+k+1)}(y)\right\}_0^\eta\right.$$

$$\left.+ (-1)^M\int_0^\eta \Phi_M(y)\left(\frac{d}{dy}\right)^m L_n^{(\alpha+k+1)}(y)\,dy\right]$$

$$= M + (-1)^M N, \text{ say.} \qquad \ldots(5.2)$$

On account of the relation

$$\frac{d}{dx}L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x)$$

we have

$$M = (A_n^k)^{-1}\left[\sum_{h=1}^{m}(-1)^{h-1}\,\Phi_h(y)\,L_{n-h+1}^{(\alpha+k+h)}(y)\right]_0^\eta$$

$$= (A_n^k)^{-1}\sum_{h=1}^{m}(-1)^{h-1}\,\Phi_h(\eta)\,L_{n-h+1}^{(\alpha+k+h)}(\eta). \qquad \ldots(5.3)$$

Using (7.6.8) from Szegö (1958), we have

$$(A_n^k)^{-1}L_{n-h+1}^{(\alpha+k+h)}(\eta) = o(n^{-k})\,o(n^{(2\alpha+2k+2h-1)/4})$$

$$= o(n^{(2\alpha-2k+2h-1)/4})$$

$$= o(n^{(h-p-1)/2})$$

$$= o(n^{(2\alpha-2k+2p+1)/4}) \text{ if } h < p+1. \qquad \ldots(5.4)$$

Therefore, we get

$$M = o(n^{(2\alpha-2k+2p+1)/4}). \qquad ...(5.5)$$

Let $p = m + \delta$, where $0 < \delta < 1$, and let

$$\left(\frac{d}{dy}\right)^m L_n^{(\alpha+k+1)}(y) \equiv S_n^{(m)}(y).$$

We consider

$$\int_0^\eta \Phi_p(y)\, S_n^{(p)}(y)\, dy = \int_0^\eta \Phi_p(y)\, dy\, \frac{1}{(1-\delta)^{1/2}} \int_0^y (y-u)^{-\delta}\, S_n^{(m+1)}(u)\, du$$

$$= \frac{1}{(1-\delta)^{1/2}} \int_0^\eta S_n^{(m+1)}(u)\, du \int_0^u (u-y)^{-\delta}\, \Phi_p(y)\, dy$$

$$= \frac{1}{(1-\delta)^{1/2}} \int_0^\eta S_n^{(M+1)}(u)\, du \left[\int_0^u (u-y)^{-\delta}\, dy\, \frac{1}{\delta^{1/2}} \int_0^y (y-t)^{\delta-1} \Phi_M(t)\, dt\right]$$

$$= \frac{1}{\delta^{1/2}(1-\delta)^{1/2}} \int_0^\eta S_n^{(M+1)}(u)\, du \left[\int_0^u \Phi_M(t)\, dt \int_t (u-y)^{-\delta} (y-t)^{\delta-1}\, dy\right]$$

$$= \int_0^\eta S_n^{(M+1)}(u)\, du \int_0^u \Phi_M(t)\, dt$$

$$= \int_0^\eta \Phi_{M+1}(u)\, du\, S_n^{(m+1)}(u)\, du$$

$$= [\Phi_{M+1}(u)\, S_n^{(m)}(u)]_0^\eta - \int_0^\eta \Phi_M(u)\, S_n^{(m)}(u)\, du$$

$$= \Phi_{M+1}(\eta)\, S_n^{(m)}(\eta) - \int_0^\eta \phi_m(u)\, S_n^{(m)}(u)\, du. \qquad ...(5.6)$$

Therefore, rearranging (5.6), we have

$$\int_0^\eta \Phi_m(u)\, S_n^{(m)}(u)\, du = \Phi_{m+1}(\eta)\, S_n^{(m)}(\eta) - \int_0^\eta \Phi_p(u)\, S_n^{(p)}(u)\, du. \qquad ...(5.7)$$

Using (5.7) in (5.2), we get

$$N = (A_n^k)^{-1} \int_0^\eta \Phi_m(y)\, S_n^{(m)}(y)\, dy$$

(*equation continued on p.* 1495)

$$= (A_n^k)^{-1} [\Phi_{m+1}(\eta) S_n^{(m)} (\eta) - \int_0^\eta \Phi_p(y) S_n^{(p)} (y) dy$$

$$= o(n^{(2\alpha-2k+2p+1)/4}) - (A_n^k)^{-1} \int_0^\eta \Phi_p(y) S_n^{(p)} (y) dy$$

following the arguments in (5.4), since $p > m$. Now, let us consider

$$(A_n^k)^{-1} \int_0^\eta \phi_p(y) S_n^{(p)} (y) dy = \{A_n^k (p+1)^{1/2}\}^{-1} \int_0^\eta y^p \phi_p(y) S_n^{(p)} (y) dy$$

$$= \int_0^{c/n} + \int_{c/n}^{\delta} = J_1 + J_2.$$

Using (7.6.8) from Szegö (1958) and (4.1), we have

$$J_1 = o(n^{-k}) \int_0^{c/n} y^p \mid \phi_p(y) \mid \; \mid L_n^{(\alpha+k+1)-p} (y) \mid dy$$

$$= o(n^{\alpha+p+1}) \, . \, n^{-p} \int_0^{c/n} \mid \phi_p(y) \mid dy$$

$$= o(n^{(2\alpha+2p-2k+1)/4}) (\log n)^\epsilon \qquad \text{...(5.8)}$$

Again, using (3.1), we get

$$J_2 = o(n^{-k}) \, o(n^{(2\alpha+2k+2p+1)/4}) \int_{c/n}^{\eta} y^p \, . \, y^{-(2\alpha+2k+2p+3)/4} \mid \phi_p(y) \mid dy$$

$$= o(n^{(2\alpha-2k+2p+1)/4}) (\log n)^\epsilon. \qquad \text{...(5.9)}$$

Using (7.6.8) from Szegö (1958), we have

$$I_2 = o(n^{-k}) \int_\eta^1 \mid \phi(y) \mid . \mid L_n^{(\alpha+k+1)} (y) \mid dy$$

$$= o(n^{-k}) \, . \, o(n^{(2\alpha+2k+1)/4}) \int_\eta^1 y^{-(2\alpha+2k+3)/4} \mid \phi(y) \mid dy$$

$$= o(n^{(2\alpha-2k+1)/4})$$

$$= o(n^{(2\alpha+2p-2k+1)/4}). \qquad \text{...(5.10)}$$

Finally, putting $\lambda - \frac{1}{3} = (2\alpha + 2k + 2p + 1)/4$ in (8.91.6) and (8.91.7) from Szegö (1958) and using the condition (3.2), we get

$$I_3 = o(n^{-k}) \int_1^\infty e^{y/2} y^{-(6\alpha+6k+6p+7)/12} n^{(2\alpha+2k+2p+1)/4} \mid \phi(y) \mid dy$$

(*equation continued on p.* 1496)

$$= o\left(n^{(2\alpha-2k+2p+1)/4}\right) \int_1^\infty e^{y/2} y^{-(6\alpha+6k+6p+7)/12} \mid \phi(y) \mid dy$$

$$= o\left(n^{(2\alpha-2k+2p+1)/4}\right). \qquad \ldots(5.11)$$

Thus the theorem is proved.

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