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FINITE SUM REPRESENTATIONS OF PARTIAL DERIVATIVES OF THE H-FUNCTION OF TWO AND MORE VARIABLES WITH RESPECT TO PARAMETERS

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Recently Buschman (1974b) and Deshpande (1978) have obtained some results for partial derivatives of special functions of one and two variables. In this paper these results are generalized for H-function of two variables. While establishing the results, the use is made of simple identities involving the Γ and ψ functions. The results established for H-function of two variables are further generalized for H-function of several variables defined by Srivastava and Panda (1976).

1. INTRODUCTION

In recent papers Buschman (1974a, b) and Deshpande (1978) have obtained some results for the partial derivatives of the special functions with respect to their parameters. In this paper these results are further generalized for the H-function of two variables (Mittal and Gupta 1972, p. 117). The H-function of two variables is a generalization of the G-function of two variables (Agarwal 1965, p. 537), Kampé de Fériet's double hypergeometric function (Appell and Kampé de Fériet 1926, p. 150) and several other special functions. Hence the formulas established here are general in character, and a large number of special cases can be obtained from them.

The results established for the H-function of two variables are further generalized for the H-function of several variables defined by Srivastava and Panda [1976, p. 271, eqn. (4.1) *et seq.*]

Throughout this paper, we use the following notation. Let $(\pm a, A)$ denote the pair of parameters (a, A) and $(-a, A)$, and let $(a, A)^2$ denote the pair of parameters (a, A) and (a, A). Again, let $\Gamma(\pm a + A)$ denote the product of two gamma functions $\Gamma(a + A)$ and $\Gamma(-a + A)$.

We take the definition and notation for the H-function of two variables due to Mittal and Gupta [1972, p. 117] as follows [see Srivastava and Panda 1976, p. 266, eqn. (1.5)]:

$$H^{0,h:(m,n);(r,u)}_{k,l:[p,q];[v,w]}\left[\begin{array}{c|l} x & ((e_k;\, E_k,\, E'_k)) : ((a_p,\, A_p));\ ((c_v,\, C_v)); \\ y & ((f_l;\, F_l,\, F'_l)) : ((b_q,\, B_q));\ ((d_w,\, {}_wD)); \end{array}\right]$$

$$= \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \theta(s)\, \theta_1(t)\, \theta_2(s,t)\, x^s y^t ds\, dt \qquad ...(1.1)$$

where $i = \sqrt{-1}$

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j s) \prod_{j=n+1}^{p} \Gamma(a_j - A_j s)} \qquad ...(1.2)$$

$$\theta_1(t) = \frac{\prod_{j=1}^{r} \Gamma(d_j - D_j t) \prod_{j=1}^{u} \Gamma(1 - c_j + C_j t)}{\prod_{j=r+1}^{w} \Gamma(1 - d_j + D_j t) \prod_{j=u+1}^{v} \Gamma(c_j - C_j t)} \qquad ...(1.3)$$

and

$$\theta_2(s,t) = \frac{\prod_{j=1}^{h} \Gamma(1 - e_j + E_j s + E'_j t)}{\prod_{j=h+1}^{k} \Gamma(e_j - E_j s - E'_j t) \prod_{j=1}^{l} \Gamma(1 - f_j + F_j s + F'_j t)} \qquad ...(1.4)$$

where an empty product is interpreted as 1, the integers $h, k, l, m, n, p, q, r, u, v, w$ are such that $0 \leqslant h \leqslant k$, $l \geqslant 0$, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, $0 \leqslant r \leqslant w$, $0 \leqslant u \leqslant v$, the coefficients E_j, E'_j, F_j, F'_j as also A_j, B_j, C_j, D_j, are all positive and the sequences of parameters (a_p), (b_q), (c_v), (d_w), (e_k) and (f_l) are so restricted that none of the poles of the integrand coincide. The contour L_1 in the complex s-plane, and the contour L_2 in the complex t-plane, are of the Mellin-Barnes type with indentations, if necessary, to ensure that they separate one set of poles from the other. Further, if we put

$$\Omega_1 = \Omega + \sum_{j=1}^{h} E_j - \sum_{j=h+1}^{k} E_j - \sum_{j=1}^{l} F_j \qquad ...(1.5)$$

$$\Omega_2 = \sum_{j=1}^{u} C_j - \sum_{j=u+1}^{v} C_j + \sum_{j=1}^{r} D_j - \sum_{j=r+1}^{w} D_j + \sum_{j=1}^{h} E'_j - \sum_{j=h+1}^{k} E'_j - \sum_{j=1}^{l} F'_j \qquad ...(1.6)$$

where

$$\Omega = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0 \qquad ...(1.7)$$

then for $\Omega_j > 0$, $j = 1, 2$, and with the points $x = 0$ and $y = 0$ being tacitly excluded, the double integral in (1.1) converges absolutely and defines an analytic function of two complex variables x and y inside the sectors given by

$$|\arg(x)| < \tfrac{1}{2}\Omega_1\pi, \; |\arg(y)| < \tfrac{1}{2}\Omega_2\pi.$$

The following formulas for the H-function of two variables will be required in our further investigation.

$$x^{\sigma}y^{\sigma'} H^{0,h:(m,n);(r,u)}_{k,l:[p,q];[v,w]}\left[\begin{array}{c|l} x & ((e_k;\, E_k, E'_k)) : ((a_p, A_p)); ((c_v, C_v)) \\ y & ((f_l;\, F_l, F'_l)) : ((b_q, B_q)); ((d_w, D_w)) \end{array}\right]$$

$$= H^{0,h:(m,n);(r,u)}_{k,l:[p,q];[v,w]}\left[\begin{array}{c|l} x & ((e_k + \sigma E_k + \sigma' E'_k;\, E_k, E'_k)) : \\ y & ((f_l + \sigma F_l + \sigma' F'_l;\, F_l, F'_l)) : \end{array}\right.$$

$$\left.\begin{array}{l} ((a_p + \sigma A_p, A_p)); ((c_v + \sigma' C_v;\, C_v)) \\ ((b_q + \sigma B_q, B_q)); ((d_w + \sigma' D_W;\, D_W)) \end{array}\right]. \qquad \ldots(1.8)$$

For $x = \lambda^B$ and $y = \mu^D$, we have

$$(-\lambda)^r(-\mu)^s \frac{\partial^r}{\partial\lambda^r}\frac{\partial^s}{\partial\mu^s} H^{0,h:(m,n);(r',u)}_{k,l:[p,q];[v,w]}$$

$$\times\left[\begin{array}{c|l} x & ((e_k;\, E_k, E'_k)) : ((a_p, A_p)); ((c_v, C_v)) \\ y & ((f_l;\, F_l, F'_l)) : ((b_q, B_q)); ((d_w, D_W)) \end{array}\right]$$

$$= H^{0,h:(m+1,n);(r'+1,u)}_{k,l:[p+1,q+1];[v+1,w+1]}\left[\begin{array}{c|l} x & ((e_k;\, E_k, E'_k)) : ((a_p, A_p)), \\ y & ((f_l;\, F_l, F'_l)) : (r, B), \end{array}\right.$$

$$\left.\begin{array}{l} (0, B); ((c_v, C_v)), (0, D) \\ ((b_q, B_q)); (s, D), ((d_w, D_w)) \end{array}\right] \qquad \ldots(1.9)$$

and for $x = \lambda^{-A}$ and $y = \mu^{-C}$, we have

$$(-\lambda)^r(-\mu)^s \frac{\partial^r}{\partial\lambda^r}\frac{\partial^s}{\partial\mu^s} H^{0,h:(m,n);(r',u)}_{k,l:[p,q];[v,w]}$$

$$\left[\begin{array}{c|l} x & ((e_k;\, E_k, E'_k)) : ((a_p, A_p)); ((c_v, C_v)) \\ y & ((f_l;\, F_l, F'_l)) : ((b_q, B_q)); ((d_w, D_w)) \end{array}\right]$$

$$= H^{0,h:(m,n+1);(r',u+1)}_{k,l:[p+1,q+1];[v+1,w+1]}\left[\begin{array}{c|l} x & ((e_k;\, E_k, E'_k)) : (1-r, A), \\ y & ((f_l;\, F_l, F'_l)) : ((b_q, B_q)), \end{array}\right.$$

$$\left.\begin{array}{l} ((a_p, A_p)); (1-s, C), ((c_v, C_v)) \\ (1, A); ((d_w, D_w)), (1, C) \end{array}\right]. \qquad \ldots(1.10)$$

In eqn. (30) of Erdelyi *et al.* (1953, p. 19), if we put $z = N$ where N is a non-negative integer, we have

$$\psi(a+N) - \psi(a) = \sum_{k=1}^{N} \frac{(-1)^{k-1}\, N!\, \Gamma(a)}{K(N-k)!\, \Gamma(a+k)} \qquad \ldots(1.11)$$

where the function $\psi(z)$ is given by $\psi(z) = \frac{d}{dz} \log \Gamma(z)$.

2. Main Formulas

In this section, we establish the following formulas for the partial derivatives of the H-function of two variables with respect to parameters. These partial derivatives are expressed as a finite sum.

$$\frac{\partial}{\partial \nu_1} \frac{\partial}{\partial \nu_2} H^{0,h:(m,n+2);(r,u+2)}_{k,l:[p+2,q];[v+2,w]} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} ((e_k;\, E_k,\, E'_k)) : \left(1-\sigma \pm \frac{\nu_1}{2}, A\right), ((a_p, A_p)); \left(1-\sigma' \pm \frac{\nu_2}{2}, C\right), ((c_v, C_v)) \\ ((f_l;\, F_l,\, F'_l)) : ((b_q, B_q)); ((d_w, D_w)) \end{matrix} \right]_{\substack{\nu_1 = N_1 \\ \nu_2 = N_2}}$$

$$= \frac{N_1!\, N_2!}{4} \sum_{l_1=0}^{N_1-1} \sum_{l_2=0}^{N_2-1} \frac{1}{l_1!\, l_2!\, (N_1 - l_1)(N_2 - l_2)}$$

$$\times H^{0,h:(m,n+2);(r,u+2)}_{k,l:[p+2,q];[v+2,w]} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} ((e_k;\, E_k,\, E'_k)) : \left(1-\sigma+\frac{N_1}{2}, A\right), \left(1-\sigma+\frac{N_1}{2}-l_1, A\right), ((a_p, A_p)); \left(1-\sigma'+\frac{N_2}{2}, C\right), \left(1-\sigma'+\frac{N_2}{2}-l_2, C\right), ((c_v, C_v)) \\ ((f_l;\, F_l,\, F'_l)) : ((b_q, B_q)); ((d_w, D_W)) \end{matrix} \right] \qquad \ldots(2.1)$$

$$\frac{\partial}{\partial \nu_1} H^{0,h:(m,n+2);(r,u)}_{k,l:[p+2,q];[v,w]} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} ((e_k;\, E_k,\, E'_k)) : \left(1-\sigma \pm \frac{\nu_1}{2}, A\right), ((a_p, A_p)); ((c_v, C_v)) \\ ((f_l;\, F_l,\, F'_l)) : ((b_q, B_q)); ((d_w, D_w)) \end{matrix} \right]_{\nu_1 = N_1}$$

(*equation continued on p.* 1518)

$$= \frac{N_1!}{2} \sum_{l_1=0}^{N_1-1} \frac{1}{l_1!\,(N_1-l_1)} H^{0,h:(m,n+2);(r,u)}_{k,l:[p+2,q];[v,w]} \left[\begin{array}{c|l} x & ((e_k;\ E_k,\ E'_k)) : \left(1-\sigma+\frac{N_1}{2},\ A\right), \left(1-\sigma+\frac{N_1}{2}-l_1,\ A\right), ((a_p,\ A_p)); ((c_v,\ C_v)) \\ y & ((f_l;\ F_l,\ F'_l)) : ((b_q,\ B_q)); ((d_w,\ D_w)) \end{array}\right] \quad \ldots(2.2)$$

$$\frac{\partial}{\partial \nu_1} \frac{\partial}{\partial \nu_2} H^{0,h:(m+2,n);(r+2,u)}_{k,l:[p,q+2];[v,w+2]} \left[\begin{array}{c|l} x & ((e_k;\ E_k,\ E'_k)) : ((a_p,\ A_p)); ((c_v,\ C_v)) \\ y & ((f_l;\ F_l,\ F'_l)) : \left(\sigma \pm \frac{\nu_1}{2}, B\right), ((b_q,\ B_q)); \left(\sigma' \pm \frac{\nu_2}{2}, D\right), ((d_w,\ D_W)) \end{array}\right]\Bigg|_{\substack{\nu_1=N_1 \\ \nu_2=N_2}}$$

$$= \frac{N_1!\,N_2!}{4} \sum_{l_1=0}^{N_1-1} \sum_{l_2=0}^{N_2-1} \frac{1}{l_1!\,l_2!\,(N_1-l_1)(N_2-l_2)}$$

$$\times H^{0,h:(m+2,n);(r+2,u)}_{k,l:[p,q+2];[v,w+2]} \left[\begin{array}{c|l} x & ((e_k;\ E_k,\ E'_k)) : ((a_p,\ A_p)); ((c_v,\ C_v)) \\ y & ((f_l;\ F_l,\ F'_l)) : \left(\sigma-\frac{N_1}{2}, B\right), \left(\sigma-\frac{N_1}{2}+l_1, B\right), ((b_q,\ B_q)); \left(\sigma'-\frac{N_2}{2},\ D\right), \left(\sigma'-\frac{N_2}{2}+l_2,\ D\right), ((d_w,\ D_w)) \end{array}\right] \quad \ldots(2.3)$$

and

$$\frac{\partial}{\partial \nu_1} H^{0,h:(m+2,n);(r,u)}_{k,l:[p,q+2];[v,w]} \left[\begin{array}{c|l} x & ((e_k;\ E_k,\ E'_k)) : ((a_p,\ A_p)); ((c_v,\ C_v)) \\ y & ((f_l;\ F_l,\ F'_l)) : \left(\sigma \pm \frac{\nu_1}{2},\ B\right), ((b_q,\ B_q)); ((d_w,\ D_W)) \end{array}\right]\Bigg|_{\nu_1=N_1}$$

$$= \frac{N_1!}{2} \sum_{l_1=0}^{N_1-1} \frac{1}{l_1!\,(N_1-l_1)} H^{0,h:(m+2,n);(r,u)}_{k,l:[p,q+2];[v,w]} \left[\begin{array}{c|l} x & ((e_k;\ E_k,\ E'_k)) : ((a_p,\ A_p)); ((c_v,\ C_v)) \\ y & ((f_l;\ F_l,F'_l)) : \left(\sigma-\frac{N_1}{2},\ B\right), \left(\sigma-\frac{N_1}{2}+l_1, B\right), ((b_q,\ B_q)); ((d_w,\ D_W)) \end{array}\right]. \quad \ldots(2.4)$$

Proof of 2.1 — Using the chain rule of derivatives and then employing the formula (1.11)), we obtain

$$\frac{\partial}{\partial \nu_1} \Gamma\left(\sigma \pm \frac{\nu_1}{2} + As\right)\Big|_{\nu_1 = N_1}$$

$$= \tfrac{1}{2}\Gamma\left(\sigma \pm \frac{N_1}{2} + As\right)\left[\psi\left(\sigma + \frac{N_1}{2} + As\right) - \psi\left(\sigma - \frac{N_1}{2} + As\right)\right]$$

$$= \tfrac{1}{2}\Gamma\left(\sigma \pm \frac{N_1}{2} + As\right) \sum_{j=1}^{N_1} \frac{(-1)^{j-1}\, N_1!\, \Gamma\left(\sigma - \frac{N_1}{2} + As\right)}{j(N_1 - j)!\, \Gamma\left(\sigma - \frac{N_1}{2} + j + As\right)} \quad \ldots(2.5)$$

Similarly, we can show that

$$\frac{\partial}{\partial \nu_2} \Gamma\left(\sigma' \pm \frac{\nu_2}{2} + Ct\right)\Big|_{\nu_2 = N_2}$$

$$= \tfrac{1}{2}\Gamma\left(\sigma' \pm \frac{N_2}{2} + Ct\right) \sum_{k=1}^{N_2} \frac{(-1)^{k-1}\, N_2!\, \Gamma\left(\sigma' - \frac{N_2}{2} + Ct\right)}{k(N_2 - k)!\, \Gamma\left(\sigma' - \frac{N_2}{2} + k + Ct\right)} \quad \ldots(2.6)$$

Now starting with the left-hand side of (2.1), expressing the H-function of two variables as a contour integral with the help of (1.1), making use of (2.5) above, the left-hand side of (2.1) can be written on some simplification as

$$\frac{N_1!\,N_2!}{4} \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} \frac{(-1)^{j+k-2}}{j\,.\,k\,.\,(N_1 - j)!(N_2 - k)!}\; H^{0,h:(m,n+3);(r,u+3)}_{k,l:[p+3,q+1];[v+3,w+1]}$$

$$\left[\begin{array}{c|l} x & ((e_k; E_k, E'_k)) : \left(1 - \sigma \pm \frac{N_1}{2}, A\right), \left(1 - \sigma + \frac{N_1}{2}, A\right), ((a_p, A_p)); \\ y & ((f_l; F_l, F'_l)) : ((b_q, B_q)), \left(1 - \sigma + \frac{N_1}{2} - j, A\right); \end{array}\right.$$

$$\left.\begin{array}{l} \left(1 - \sigma' \pm \frac{N_2}{2}, C\right), \left(1 - \sigma' + \frac{N_2}{2}, C\right), ((c_v, C_v)) \\ ((d_w, D_w)), \left(1 - \sigma' + \frac{N_2}{2} - k, C\right) \end{array}\right]. \quad \ldots(2.7)$$

Now consider only the H-function in (2.7). Employing the formulas (1.8), (1.10) and again (1.8), it can be written as

$$(-1)^{N_1 + N_2 - j - k}\, (\lambda_1)^{(N_1/2)+\sigma} (\lambda_2)^{(N_2/2)+\sigma'} .\; \frac{\partial^{N_1 - j}\, \partial^{N_2 - k}}{\partial \lambda_1^{N_1 - j}\, \partial \lambda_2^{N_2 - k}} \times$$

(*equation continued on p.* 1520)

$$\times \{\lambda_1^{(N_1/2)-\sigma-j}\,\lambda_2^{(N_2/2)-\sigma'-k}\,H^{0,h:(m,n+2);(r,u+2)}_{k,l:[p+2,q];[v+2,w]}\left[\begin{array}{c|l} x & ((e_k;\,E_k,\,E'_k)): \\ y & ((f_l;\,F_l,\,F'_l)): \end{array}\right.$$

$$\left.\begin{array}{ll} \left(1-\sigma+\frac{N_1}{2},\,A\right)^2,\,((a_p,\,A_p)); & \left(1-\sigma'+\frac{N_2}{2},\,C\right)^2,\,((c_v,\,C_v)) \\ ((b_q,\,B_q)); & ((d_w,\,D_w)) \end{array}\right] \quad \ldots(2.8)$$

where $x=\lambda_1^{-A}$, and $y=\lambda_2^{-C}$.

Now employing the formula

$$\frac{\partial^{N-j}}{\partial\lambda^{N-j}}\left\{\lambda^{-j}\,.\,\lambda^{(N/2)-\sigma}\,H\left[\begin{matrix}\lambda^{-A}\\ y\end{matrix}\right]\right\}$$

$$=\sum_{l=0}^{N-j}\binom{N-j}{l}\frac{\partial^{N-j-l}}{\partial\lambda^{N-j-l}}(\lambda^{-j})\,\frac{\partial^l}{\partial\lambda^l}\left\{\lambda^{(N/2)-\sigma}H\left[\begin{matrix}\lambda^{-A}\\ y\end{matrix}\right]\right\}$$

(2.8) becomes on some simplification as

$$\sum_{l_1=0}^{N_1-j}\sum_{l_2=0}^{N_2-k}\frac{(N_1-j)!\,(N_2-k)!\,\Gamma(N_1-l_1)\,\Gamma(N_2-l_2)}{l_1!\,l_2!\,(N_1-j-l_1)!\,(N_2-k-l_2)!\,\Gamma(j)\,\Gamma(k)}$$

$$\times\,H^{0,h:(m,n+2);(r,u+2)}_{k,l:[p+2,q];[v+2,w]}\left[\begin{array}{c|l} x & ((e_k;\,E_k,\,E'_k)):\left(1-\sigma+\frac{N_1}{2},\,A\right), \\ y & ((f_l;\,F_l,\,F'_l)): \end{array}\right.$$

$$\begin{array}{l}\left(1-\sigma+\frac{N_1}{2}-l_1,\,A\right),\,((a_p,\,A_p));\,\left(1-\sigma'+\frac{N_2}{2},\,C\right),\\ ((b_q,\,B_q));\end{array}$$

$$\left.\begin{array}{l}\left(1-\sigma'+\frac{N_2}{2}-l_2,\,C\right),\,((c_v,\,C_v))\\ ((d_w,\,D_w))\end{array}\right]. \quad \ldots(2.9)$$

Making use of (2.9) in (2.7), interchanging the order of summations in $\sum\limits_{j=1}^{N_1}\sum\limits_{l_1=0}^{N_1-j}$, $\sum\limits_{k=1}^{N_2}\sum\limits_{l_2=0}^{N_2-k}$ and inner sums then simplified to 1, we arrive at the desired result (2.1).

Similar is the proof of (2.2).

Proofs of (2.3) and (2.4) are on the same lines as that of (2.1) and (2.2) except that we have to use the result (1.9) instead of (1.10) in deriving these results.

3. Special Cases

(I) In (2.1), taking all the A's, B's, C's, D's, E's and F's equal to 1, the H-function of two variables reduces to the G-function of two variables and we obtain

$$\frac{\partial}{\partial \nu_1}\frac{\partial}{\partial \nu_2} G^{0,h:(m,n+2);(r,u+2)}_{k,l:[p+2,q];[v+2,w]}\left[\begin{array}{c|l} x & (e_k) : 1-\sigma \pm \frac{\nu_1}{2}, (a_p); 1-\sigma' \pm \frac{\nu_2}{2}, (c_v) \\ y & (f_l) : \qquad (b_q); \qquad (d_w) \end{array}\right]\Bigg|_{\substack{\nu_1=N_1 \\ \nu_2=N_2}}$$

$$= \frac{N_1!\,N_2!}{4}\sum_{l_1=0}^{N_1-1}\sum_{l_2=0}^{N_2-1}\frac{1}{l_1!\,l_2!\,(N_1-l_1)(N_2-l_2)}$$

$$\times G^{0,h:(m,n+2);(r,u+2)}_{k,l:[p+2,q];[v+2,w]}\left[\begin{array}{c|l} x & (e_k) : 1-\sigma+\frac{N_1}{2}, 1-\sigma+\frac{N_1}{2}-l_1, (a_p); 1-\sigma'+\frac{N_2}{2}, 1-\sigma'+\frac{N_2}{2}-l_2, (c_v) \\ y & (f_l) : \qquad (b_q); \qquad (d_w) \end{array}\right]. \qquad ...(3.1)$$

The G-function of two variables obtained above is a slight variant of the one introduced earlier by Agarwal (1965, p. 537), though in essence the function remains the same. [See also Srivastava and Panda (1976), p. 267, eqn. (1.11).]

(II) In (2.1), if we put $h = k = l = 0$, we obtain on simplification

$$\frac{\partial}{\partial \nu} H^{m,n+2}_{p+2,q}\left[x\left|\begin{array}{l}\left(1-\sigma\pm\frac{\nu}{2}, A\right), ((a_p, A_p)) \\ ((b_q, B_q))\end{array}\right.\right]\Bigg|_{\nu=N}$$

$$= \frac{N!}{2}\sum_{l=0}^{N-1}\frac{1}{l!\,(N-l)} H^{m,n+2}_{p+2,q}\left[x\left|\begin{array}{l}\left(1-\sigma+\frac{N}{2}, A\right), \left(1-\sigma+\frac{N}{2}-l, A\right), ((a_p, A_p)) \\ ((b_q, B_q))\end{array}\right.\right] \qquad ...(3.2)$$

which does not seem to have been noticed earlier.

(III) In (2.1), taking $k = h$, $m = r = l$, $p = n$, $v = u$, E_k, E'_k, ..., all equal to unity, $b_1 = d_1 = 0$, and making use of the result [Srivastava and Panda 1976, p. 272, eqn. (4.7)], we have a known result due to Deshpande (1978) as

$$\frac{\partial}{\partial\nu_1}\frac{\partial}{\partial\nu_2}\left\{\Gamma\left(\sigma\pm\frac{\nu_1}{2}\right)\Gamma\left(\sigma'\pm\frac{\nu_2}{2}\right)F^{A,B+2,B'+2}_{C,D,D'}\right.$$
$$\left.\times\left[\begin{array}{c|c} x & (a):\sigma\pm\frac{\nu_1}{2},(b);\sigma'\pm\frac{\nu_2}{2},(b') \\ y & (c):\qquad (d);\qquad (d') \end{array}\right]\right|_{\substack{\nu_1=N_1\\ \nu_2=N_2}}$$

$$=\frac{N_1!\,N_2!}{4}\sum_{l_1=0}^{N_1-1}\sum_{l_2=0}^{N_2-1}\frac{\Gamma\left(\sigma-\frac{N_1}{2}\right)\Gamma\left(\sigma'-\frac{N_2}{2}\right)\Gamma\left(\sigma-\frac{N_1}{2}+l_1\right)\Gamma\left(\sigma'-\frac{N_2}{2}+l_2\right)}{l_1!\,l_2!\,(N_1-l_1)\,(N_2-l_2)}$$
$$\times F^{A,B+2,B'+2}_{C,D,D'}\left[\begin{array}{c|c} x & (a):\sigma-\frac{N_1}{2},\sigma-\frac{N_1}{2}+l_1,(b);\sigma'-\frac{N_2}{2},\sigma'-\frac{N_2}{2}+l_2,(b'); \\ y & (c):\qquad (d);\qquad (d'); \end{array}\right]. \quad \ldots(3.3)$$

In (3.3) above, if we put $A = B = B' = D = D' = 0$, and $C = 1$, we have a known formula (Deshpande 1978, for the partial derivatives of the Appell function F_3 (Appell and Kampé de Fériet 1926, p. 14) which can further be reduced to the known result of Buschman [1974a, (33), p. 824].

(IV) In (2.3) and (2.4), if we put $h = k = l = 0$, and $\sigma = 0$ we obtain a known result due to Buschman [1974b, (22), p. 153]. Proceeding on the same lines as above, special cases of (2.2) to (2.4) can be obtained. The details are omitted to save space.

4. Further Generalization

If we use the abbreviations, definitions and notations of Srivastava and Panda [1976, p. 271, eqn. (4.1)] for the H-function in several complex variables $x_1, \ldots, x_n$, defined by them, we can generalize our formulas (2.1) to (2.4) for the H-function of n variables. One such result is mentioned below. Other results can easily be obtained.

$$\prod_{j=1}^{n}\frac{\partial}{\partial\nu^{(j)}}\,H^{0,\lambda:(\mu^{(1)},\beta^{(1)}+2),\ldots,(\mu^{(n)},\beta^{(n)}+2)}_{A,C:[B^{(1)}+2,D^{(1)}];\ldots;[B^{(n)}+2,D^{(n)}]}\left[\begin{array}{c|c} x_1,\ldots,x_n & [(a):\theta^{(1)},\ldots,\theta^{(n)}]: \\ & [(c):\psi^{(1)},\ldots,\psi^{(n)}]: \end{array}\right.$$
$$\left[1-\sigma^{(1)}\pm\frac{\nu^{(1)}}{2}:\alpha^{(1)}\right],[(b^{(1)}):\phi^{(1)}];\ldots;\left[1-\sigma^{(n)}\pm\frac{\nu^{(n)}}{2}:\alpha^{(n)}\right],$$
$$[(d^{(1)}):\delta^{(1)}];\ldots;$$
$$\left.\left.\begin{array}{c}[(b^{(n)}):\phi^{(n)}] \\ [(d^{(n)}):\delta^{(n)}]\end{array}\right]\right|_{\substack{\nu^{(j)}=N_j\\ j=1,\ldots,n}} =$$

(*equation continued on p.* 1523)

$$= \frac{\prod_{j=1}^{n} (N_j)!}{2^n} \sum_{l_1=0}^{N_1-1} \cdots \sum_{l_n=0}^{N_n-1} \frac{1}{\prod_{j=1}^{n} [l_j!\,(N_j - l_j)]}$$

$$\times H^{0,\lambda:(\mu^{(1)},\beta^{(1)}+2),\ldots,(\mu^{(n)},\beta^{(n)}+2)}_{A,C:[B^{(1)}+2,D^{(1)}],\ldots,[B^{(n)}+2,D^{(n)}]} \left[\begin{array}{l} [(a):\theta^{(1)},\ldots,\theta^{(n)}]: \\ [(c):\psi^{(1)},\ldots,\psi^{(n)}]: \end{array}\right.$$

$$\left[1-\sigma^{(1)}+\frac{N_1}{2}:\alpha^{(1)}\right],\left[1-\sigma^{(1)}+\frac{N_1}{2}-l_1:\alpha^{(1)}\right],[(b^{(1)}):\phi^{(1)}];\ldots;$$
$$[(d^{(1)}):\delta^{(1)}];\ldots;$$

$$\left.\begin{array}{l}\left[1-\sigma^{(n)}+\frac{N_n}{2}:\alpha^{(n)}\right],\left[1-\sigma^{(n)}+\frac{N_n}{2}-l_n:\alpha^{(n)}\right],[(b^{(n)}):\phi^{(n)}]; \\ \qquad [(d^{(n)}):\delta^{(n)}]; \end{array} x_1,\ldots,x_n\right]. \qquad \ldots(4.1)$$

The proof of (4.1) runs parallel to that of deriving (2.1), and hence we omit details.

The special cases of (4.1) can be obtained in terms of generalized Lauricella function

$$F^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}}\,[x_1,\ldots,x_n]$$

which is defined by Srivastava and Daoust (1969, p. 454), or in terms of *G*-function of *n* variables [see Srivastava and Panda 1976, p. 273, eqn. (4.9)].

$$G^{0,\lambda:(\mu^{(1)},\beta^{(1)});\ldots;(\mu^{(n)},\beta^{(n)})}_{A,C:[B^{(1)},D^{(1)}];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{array}{l}(a):(b^{(1)});\ldots;(b^{(n)}); \\ (c):(d^{(1)});\ldots;(d^{(n)});\end{array} x_1,\ldots,x_n\right).$$

To obtain such results, relationship (4.7) and (4.9) of Srivastava and Panda (1976, p. 272) can be utilized.

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