APPLICATION OF G.P.D.P. TO HEAT CONDUCTION PROBLEM

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The governing principle of dissipative processes is applied to get the variational solution of heat conduction phenomenon in a semi-infinite solid and in the plate of finite thickness. The heat flux at the boundary surface obtained from universal form of Gyarmati's principle is compared with the results obtained by exact solution and Goodman's technique. It is found that the results of universal form differ by about 2% from that of exact. Our results are quite better than that of Goodman's technique. It is also shown that the flux representation of governing principle yields exactly same result as obtained by Lagrangian thermodynamics of Biot.

INTRODUCTION

The main aim of the present analysis is to develop a new variational method to investigate temperature distribution in a semi infinite and finite solid due to linear heat conduction. The method is based on the governing principle of dissipative processes which describes the evolution of dissipative transport processes in space and time (Gyarmati 1969, 1970). It is well known that G.P.D.P. results into two partial forms: force and flux representations. It has already been proved by Gyarmati and his co-workers (1970) that the force representation is equivalent to local potential method of Glansdroff and Prigogine (1964). Recently Singh (1973, 1976a,b,c) proved the validity of this fact considering variational solution of Benard stability problem and heat transfer in a rod. The critical wave and Rayleigh numbers obtained by using the force representation of G.P.D.P. are exactly the same as obtained by local potential method (Schecter 1967, Glansdroff and Prigogine 1971).

Singh (1973, 1977) has also proved that the Lagrangian thermodynamics of Biot is equivalent to flux representation of Gyarmati's principle. In this paper we have established the validity of above fact by investigating the temperature distribution in semi-infinite solid and in a plate of finite thickness. In both problems the flux representation yields exactly the same results as obtained by Biot's Lagrangian thermodynamics (1970).

FORMULATION OF PROBLEM

The governing principle of dissipative processes in its universal form is given by

$$\delta \int_V [\sigma - \psi - \phi]\, dV = 0 \qquad \ldots(1)$$

where σ denotes the entropy production and ψ and ϕ are the dissipation potentials (Gyarmati 1970). For heat conduction phenomena in a rigid body we write the principle (1) in Fourier picture as (Gyarmati 1969)

$$\delta \int_V \left[- \mathbf{J}_q \cdot \nabla T - \frac{K}{2} \nabla T \cdot \nabla T - \frac{1}{2K} \mathbf{J}_q \cdot \mathbf{J}_q \right] dV = 0 \qquad \ldots(2)$$

Here V denotes a bounded region in 3 dimensional space, T is temperature, K is the ordinary heat conduction coefficient. The heat current density $\mathbf{J}_q$ satisfies following internal energy balance equation without source term

$$C \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{J}_q = 0 \qquad \ldots(3)$$

where C is the heat capacity of the solid.

The volume integral (2) is maximum at any instant of time for the real physical process. It is fundamentally important that maximum is zero for any instant of time (Gyarmati 1969). However, in course of approximation procedure the volume integral generally becomes a function of time and therefore it appears natural to integrate it over the time interval $0 < t < \infty$ during which process has taken place. Taking this into account principle (2) becomes

$$\delta \int_0^\infty \int_V \left[- \mathbf{J}_q \cdot \nabla T - \frac{K}{2} (\nabla T)^2 - \frac{1}{2K} \mathbf{J}_q^2 \right] dt\, dV = 0. \qquad \ldots(4)$$

In dual field method we introduce by definition an approximate temperature field T^* which satisfies the following constitutive relation (Stark 1974)

$$\mathbf{J}_q = - K \nabla T^*. \qquad \ldots(5)$$

The principle (4) with (5) becomes

$$\delta \int_0^\infty \int_V [\nabla T^* \nabla T - \tfrac{1}{2} (\nabla T)^2 - \tfrac{1}{2} (\nabla T^*)^2]\, dt\, dV = 0 \qquad \ldots(6)$$

and energy balance becomes

$$C \frac{\partial T}{\partial t} - K \nabla^2 T^* = 0. \qquad \ldots(7)$$

In the following we will apply the principle (6) to get temperature distribution in a semi-infinite solid and in a plate of thickness L.

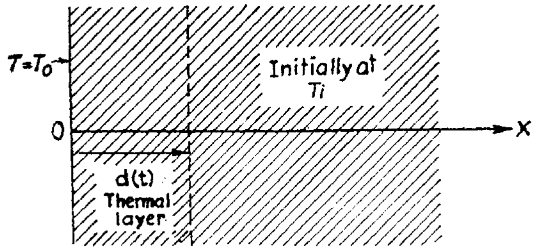

FIG. 1. A semi-infinite region and the "thermal layer".

SEMI-INFINITE SOLID

Consider a semi-infinite solid $0 \leqslant x < \infty$ which is initially at a uniform temperature T_i. For $t > 0$ the boundary surface at $x = 0$ is kept at a constant temperature T_0. Fig. 1 shows the geometry, the boundary conditions and the location of thermal layer $d(t)$ at any instant of time t. For all practical purposes the plate in the region $x \geqslant d(t)$ is at initial temperature T_i and there is no heat flux in $x \geqslant d(t)$. Since this is one dimensional problem, the principle (6) and balance equation (7) become

$$\delta \int_0^\infty \int_0^d \left[\frac{\partial T}{\partial x}\frac{\partial T^*}{\partial x} - \tfrac{1}{2}\left(\frac{\partial T}{\partial x}\right)^2 - \tfrac{1}{2}\left(\frac{\partial T^*}{\partial x}\right)^2\right] dt\, dx = 0 \qquad \ldots(8)$$

$$C\frac{\partial T}{\partial t} - K\frac{\partial^2 T^*}{\partial x^2} = 0. \qquad \ldots(9)$$

Consistent with the boundary conditions

$$T\Big|_{x=0} = T_0,\ T\Big|_{x=d} = T_i,\ \frac{dT}{dx}\Big|_{x=d} = 0 \quad \text{for } t > 0$$

we assume the temperature profile in the following form

$$\frac{T - T_i}{T_0 - T_i} = \left(1 - \frac{x}{d}\right)^2. \qquad \ldots(10)$$

The balance equation (9) with the help of (10) gives

$$\mathbf{J}_q = -\lambda\frac{\partial T^*}{\partial x} = C(T_0 - T_i)\left(\frac{1}{3} - \frac{x^2}{d^2} + \frac{2}{3}\frac{x^3}{d^3}\right)\dot{d}. \qquad \ldots(11)$$

The principle (8) becomes with (10) and (11) now

$$\delta \int_0^\infty \left[\frac{14}{30}\frac{K}{C}\dot{d} - \frac{4}{3}\frac{K^2}{C^2}\frac{1}{d} - \frac{13}{315}d\dot{d}^2\right] dt = 0. \qquad \ldots(12)$$

Euler-Lagrange equation for this variational problem is

$$d^3\ddot{d} + 0.5d^2\dot{d}^2 + 16.1538\frac{K^2}{C^2} = 0. \qquad ...(13)$$

The solution of this equation with initial condition $d = 0$ when $t = 0$ is easily obtained as

$$d = 3.371642\sqrt{\frac{K}{C}\,t}. \qquad ...(14)$$

In flux representation we take variation w.r.t fluxes only, i.e. variation with respect to forces is zero. Thus the principle (8) reduces to

$$\delta\int_0^\infty\int_0^{d_1}\left[\frac{\partial T}{\partial x}\frac{\partial T^*}{\partial x} - \tfrac{1}{2}\left(\frac{\partial T^*}{\partial x}\right)^2\right]dt\,dx = 0. \qquad ...(15)$$

which yields to

$$\delta\int_0^\infty\left\{\frac{K}{C}\left[\frac{1}{3}\frac{d\dot{d}}{d_1} - \frac{1}{10}\frac{d^2\dot{d}}{d_1^2}\right] - \tfrac{1}{2}\left[\frac{d_1}{9} + \frac{1}{5}\frac{d_1^5}{d^4} + \frac{4}{63}\frac{d_1^7}{d^6} - \frac{2}{9}\frac{d_1^3}{d^2} + \frac{1}{9}\frac{d_1^4}{d^3} - \frac{2}{9}\frac{d_1^6}{d^5}\right]\dot{d}^2\right\}dt = 0. \qquad ...(16)$$

Here d is the only variational parameter which occurs in the profile of $\mathbf{J}_q$. The variation of (16) is taken with respect to d only. Euler-Lagrange equation is now obtained with $d = d_1$ as

$$\frac{d}{dt}\left[\frac{14}{30}\frac{K}{C} - \frac{26}{315}d\dot{d}\right] = 0 \qquad ...(17)$$

with transversality condition

$$\left(\frac{14}{30}\frac{K}{C} - \frac{26}{315}d\dot{d}\right)\Big|_{t=\infty} = 0. \qquad ...(18)$$

Thus we get

$$\frac{7}{30}\frac{K}{C} - \frac{13}{315}d\dot{d} = 0. \qquad ...(19)$$

This is a first order equation and can be integrated with initial condition $d = 0$ at $t = 0$ we find

$$d = 3.36\sqrt{\frac{K}{C}\,t} \qquad ...(20)$$

which is exactly the same as obtained by Biot's Lagrangian analysis (1970).

If we use a third order polynomial expression

$$\frac{T - T_i}{T_0 - T_i} = 1 - \frac{3}{2}\frac{x}{d} + \tfrac{1}{2}\frac{x^3}{d^3} \qquad \text{...(21)}$$

for temperature; Euler-Lagrange equation of principle (8) gives

$$d^3\ddot{d} + 0.5\, d^2\dot{d}^2 + 5.8332\frac{K^2}{C^2} = 0 \qquad \text{...(22)}$$

which can be solved as before and we get

$$d = 2.61367\sqrt{\frac{K}{C}t}. \qquad \text{...(23)}$$

Similarly by using a fourth degree profile

$$\frac{T - T_i}{T_0 - T_i} = 1 - \frac{2x}{d} + 2\left(\frac{x}{d}\right)^3 - \left(\frac{x}{d}\right)^4 \qquad \text{...(24)}$$

we get

$$d^3\ddot{d} + 0.5\, d^2\dot{d}^2 + 20.5252\,\frac{K^2}{C^2} = 0 \qquad \text{...(25)}$$

which gives

$$d = 3.579691\sqrt{\frac{K}{C}t}. \qquad \text{...(26)}$$

Here we compare our results with Goodman integral method (Ozisik 1968) (see Table I). For comparison we have chosen heat flux at boundary surface $x = 0$ which is a quantity of practical interest for heat conduction problem

$$\mathbf{J}_q = -K\left.\frac{\partial T}{\partial x}\right|_{x=0} = C\sqrt{KC}\,(T_0 - T_i). \qquad \text{...(27)}$$

Heat Conduction in Finite Region

Since the region is finite we treat the problem in two stages: (1) The first stage when the thermal layer thickness is less than the slab thickness L, (2) The second stage during which d exceeds the slab thickness L. The plate is initially at temperature $T = 0$.

The first stage — The treatment is same as discussed in previous section. Using a quadratic profile

$$T = T_0\left(1 - \frac{x}{d}\right)^2 \qquad \text{...(28)}$$

we find

$$d = 3.3716\sqrt{\frac{K}{C}t} \qquad \text{...(29)}$$

TABLE I

Temperature profile	C from G.P.D.P.	C from Goodman integral method	Per cent error involved in G.P.D.P.	Per cent error involved in Goodman method	Exact
Quadratic approx. [eqn. (10)]	0.59311				
Cubic approx. [eqn. (21)]	0.5739	0.530	1.57%	6%	0.565
Fourth degree approx. [eqn. (24)]	0.5587	0.548	+ 1.1%	3%	

Equation (28) is valid in $0 \leqslant x \leqslant d$ for $d \leqslant L$ or $t \leqslant t_1$ where t_1 is defined as the time required for the thermal layer to reach the boundary surface at $x = L$. The value of t_1 is determined by substituting $d = L$ in (20) to get

$$t_1 = 1.08796 \frac{CL^2}{K}. \qquad \ldots(30)$$

The second stage — For times $t > t_1$ thickness of thermal layer exceeds the thickness of plate L, hence the concept of thermal layer loses its significance. Temperature in this phase is also assumed to be well represented by a parabolic approximation

$$T = (T_0 - q_2)\left(1 - \frac{x}{L}\right)^2 + q_2. \qquad \ldots(31)$$

Here q_2 is the unknown temperature at the boundary $x = L$ and will be determined by the principle (8).

Balance eqn. (9) with the help of (3) gives

$$\mathbf{J}_q = -\lambda \frac{\partial T^*}{\partial x} = C\dot{q}_2 \left[\frac{x^3}{3l^2} - \frac{x^2}{L} + \frac{2L}{3}\right], \qquad \ldots(32)$$

Principle (8) becomes

$$\delta \int_0^\infty \left[\frac{16}{15}(T_0\dot{q}_2 - q_2\dot{q}_2)\, CL - \frac{4K}{3L}(T_0 - q_2)^2 - \frac{68}{315}\frac{C^2L^3}{K}\dot{q}_2\right] dt = 0.$$

$$\ldots(33)$$

Euler-Lagrange equation of above variational problem is

$$\ddot{q}_2 + \frac{630}{102}\frac{K^2}{C^2L^4}(T_0 - q_2) = 0. \qquad ...(34)$$

Since q_2 is always finite and at $t = t_1$, $q_2 = 0$ solution of eqn. (34) will be

$$\frac{q_2}{T_0} = 1 - \exp\left\{- 0.218622\left(\frac{t}{t_1} - 1\right)\right\}. \qquad ...(35)$$

In flux representation we once again take variation w.r.t. fluxes only. Principle (15) in this case becomes

$$\delta \int_0^\infty \left\{\frac{32}{60} CL(T_0 - q_2^*)\, \dot{q}_2 - \frac{34}{315}\frac{C^2L^3}{K}\dot{q}_2^2\right\} dt = 0. \qquad ...(36)$$

Variation is to be taken w.r.t. unstarred q_2 and after variation we put $q_2 = q_2^*$.

Euler-Lagrange equation of (36) is

$$\frac{d}{dt}\left[\frac{32}{60} CL(T_0 - q_2) - \frac{68}{315}\dot{q}_2\right] = 0 \qquad ...(37)$$

along with transversability condition

$$\left[\frac{32}{60} CL(T_0 - q_2) - \frac{68}{315}\frac{C^2L^3}{K}\dot{q}_2\right]\Bigg|_{t=\infty} = 0. \qquad ...(38)$$

Equation (37) gives

$$\frac{32}{60} CL(T_0 - q_2) - \frac{68}{315}\frac{C^2L^3}{K}\dot{q}_2 = 0 \qquad ...(39)$$

i.e. $$\frac{4080}{10080}\frac{CL^2}{K}\dot{q}_2 + q_2 = T_0 \qquad ...(40)$$

i.e. $$4.57\, t_1 \dot{q}_2 + q_2 = T_0. \qquad ...(41)$$

We integrate above equation with initial value $t = t_1$, $q_2 = 0$. The result is

$$\frac{q_2}{T_0} = 1 - \exp\left\{- 0.218\left(\frac{t}{t_1} - 1\right)\right\} \qquad ...(42)$$

which is same as obtained by Biot's Lagrangian analysis (1970).

For comparing the result with Goodman integral method we use a cubic profile (Ozisik 1968)

$$\frac{T - T_i}{T_0 - T_i} = 1 + \eta(t)\frac{x}{L} - \frac{1}{3}\eta(t)\left(\frac{x}{L}\right)^3. \qquad ...(43)$$

Here plate is initially at temperature T_i, and $\eta(t)$ is unknown function of time which will be determined by principle. In this case from (23) we get

$$t_1 = 0.1463 \frac{CL^2}{K} \cdot$$

Substituting (43) in balance eqn. (9) we get

$$\mathbf{J}_q = -K \frac{\partial T^*}{\partial x} = C\left[\frac{3}{20}\dot{\eta}L + \frac{K}{CL} + \frac{\eta}{12}\frac{x^4}{12L^3} - \frac{\dot{\eta}x^2}{2L}\right]. \qquad \ldots(44)$$

Principle (8) after integration w.r.t. x gives

$$\delta \int_0^\infty \left[-0.076 \frac{K}{C}\eta\dot{\eta}L - \frac{1}{3}\frac{K^2}{C^2}\frac{\eta}{L} - 0.5333 \frac{K^2}{C^2}\frac{\eta^2}{L}\right.$$

$$\left. - 0.016637\dot{\eta}^2L^3 - \frac{K^2}{C^2}\frac{1}{L}\right] dt = 0. \qquad \ldots(45)$$

Euler-Lagrange equation of above variational problem is

$$L^4\ddot{\eta} - 32.5858\,\eta \frac{K^2}{C^2} = 10.18308\,\eta \frac{K^2}{C^2} \qquad \ldots(46)$$

which can be solved easily. The result is

$$\eta(t) = -3.1468\, e^{-5\cdot 7084Kt/CL^2} - 0.3124 \qquad \ldots(47)$$

$$= -3.1468\, e^{-0\cdot 8356t/t_1} - 0.3124. \qquad \ldots(48)$$

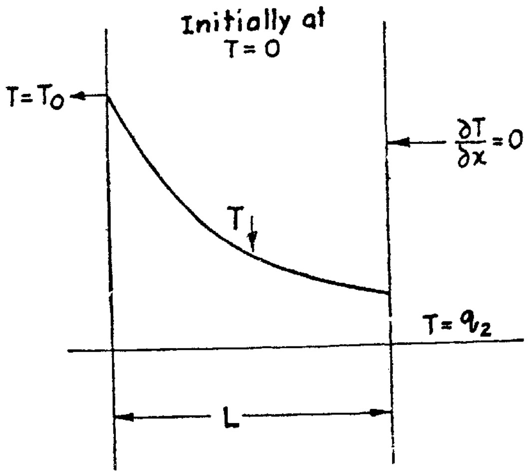

FIG. 2. Distribution of T for second stage in heating of a plate insulated at $x = L$.

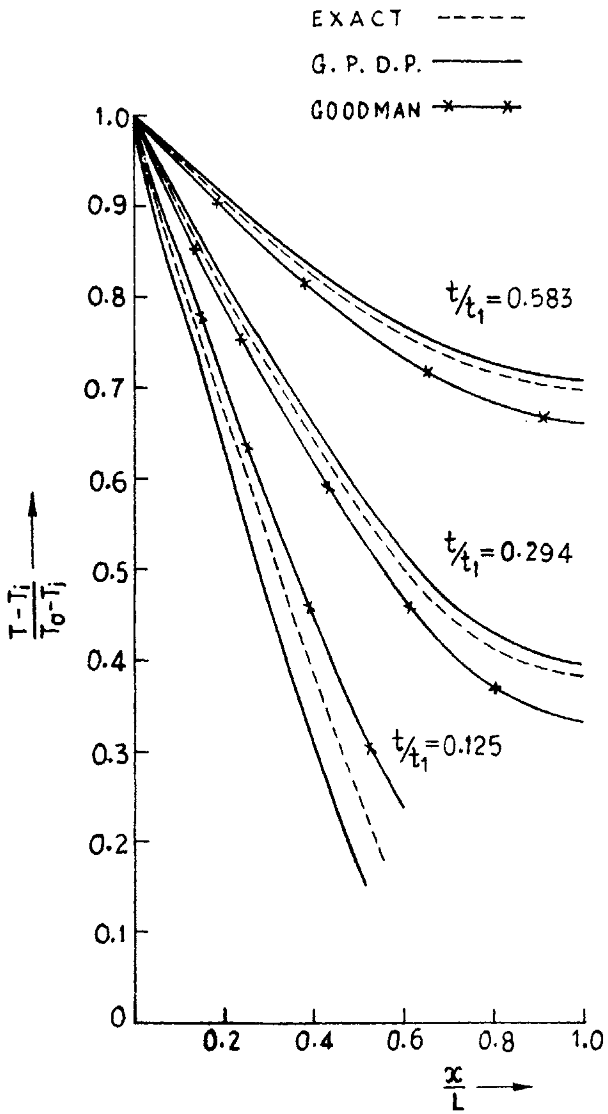

FIG. 3. Comparison of temperature distribution from exact, G.P.D.P. and Goodman integral solution for a slab of thickness L.

For three values of non-dimensional time parameter t/t_1, temperature $(T - T_i)/(T_0 - T_i)$ has been plotted against x/L. As can be seen in Fig. 3, the agreement with exact solution is quite good.

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