FLOW OF VISCOELASTIC FLUID INDUCED BY ELLIPTIC HARMONIC OSCILLATIONS OF A DISK

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This paper considers the flow of incompressible viscoelastic fluid (Rivlin-Ericksen 1955) induced by the non-torsional elliptic harmonic oscillations of an infinite rigid disk. The boundary value problem has been solved for the velocity as well as pressure distribution with the assumption that the amplitude of the oscillations is small.

1. INTRODUCTION

The study of oscillatory boundary layers in a Newtonian viscous fluid has already been discussed by several authors (Landau and Lifschitz 1966, Schlichting 1960). Siddappa and Balagondar (1975) extended their work to non-Newtonian [Rivlin-Ericksen viscoelastic (1955) model] fluids.

The object of this paper is to consider the small amplitude wave phenomenon in the flow of incompressible, viscoelastic fluid [Rivlin-Ericksen (1955) type] induced by the elliptic harmonic oscillations of an infinite rigid disk. The problem is entirely a boundary value problem and the solution for the velocity and pressure distributions are obtained.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

We take the rectangular Cartesian co-ordinates (x, y, z) so that x- and y-axes lie in the plane of the disk at $z = 0$ and z-axis normal to it. Let $u = u(z, t)$, $v = 0$, $w = 0$ be the components of the velocity along x, y, z axes respectively and $p = p(z, t)$ be the pressure.

The equations governing the unsteady flow of incompressible viscoelastic fluid (suggested by Rivlin-Ericksen), when there are no external body forces, are (Siddappa 1972)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial z^2} + \beta \frac{\partial^2}{\partial z^2}\left(\frac{\partial u}{\partial t}\right) \qquad \ldots(1)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + 2(2\beta + \gamma)\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} \qquad \ldots(2)$$

where ρ is the density and α, β, γ are respectively the kinematic viscosity, kinematic viscoelasticity and kinematic cross-viscosity.

Since the flow is induced by the non-torsional elliptic harmonic oscillations of the boundary disk at $z = 0$, we can take the following boundary conditions to solve (1) and (2) :

$$u(z, t) = Ae^{i\omega t} + Be^{-i\omega t} \text{ on } z = 0,\ t > 0 \qquad ...(3)$$

$$u(z, t) \to 0 \text{ as } z \to \infty \text{ and for all } t \qquad ...(4)$$

where A and B are complex constants and ω is the frequency of the imposed oscillations.

Solution of (2) for p is given by

$$p(z, t) = \rho(2\beta + \gamma)\left(\frac{\partial u}{\partial z}\right)^2 + g(t). \qquad ...(5)$$

We apply Laplace transform method to solve (1) subject to the initial conditions (4).

It turns out that the solution of the problem can readily be represented by the Laplace inversion integral in the form :

$$u(z, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty}\left(\frac{A}{p - i\omega} + \frac{B}{p + i\omega}\right)\exp[-\{p/(\alpha + \beta p\}^{1/2} z)]\, e^{pt} dp,$$

$$\gamma > 0.$$

Here the integrands have simple poles at $p = \pm\, i\omega$ and have branch point at $p = 0$, $p = -\,\alpha/\beta$. For sufficiently large time t, the Bromwich contour integral can be computed by inserting a branch out in the complex p-plane from the branch points along the negative real axis.

The Bromwich integrals can be evaluated by using the theory of residues. When all the singularities are simple poles, it turns out from the computation of residues that the fluid velocity $u(z, t)$ is

$$u(z, t) = Ae^{i\omega t}\,.\,\exp\left(-\,z\sqrt{i\omega/(\alpha + \beta i\omega)}\right)$$
$$+ Be^{-i\omega t}\exp\left(-\,z\sqrt{i\omega/(\beta i\omega - \alpha)}\right)$$
$$+ \frac{1}{\pi}\int_0^\infty\left(\frac{A}{x + i\omega} + \frac{B}{x - i\omega}\right)e^{-xt}\sin z\sqrt{\frac{x}{\alpha - \beta x}}\, dx. \qquad ...(6)$$

Let
$$\left(\frac{i\omega}{\alpha + \beta i\omega}\right)^{1/2} = r_1\exp(i\theta_1),\ \left(\frac{i\omega}{\beta i\omega - \alpha}\right)^{1/2} = r_2\exp(i\theta_2), \qquad ...(7)$$

then
$$\left.\begin{aligned} r_1 &= \frac{\sqrt{\omega}}{(\alpha^2 + \beta^2\omega^2)^{1/4}},\quad \theta_1 + \theta_2 = \frac{\pi}{2},\\ \theta_1 &= \tfrac{1}{2}\tan^{-1}\left(\frac{\alpha}{\beta\omega}\right),\ \theta_2 = \tfrac{1}{2}\tan^{-1}\left(\frac{-\alpha}{\beta\omega}\right). \end{aligned}\right\} \qquad ...(8)$$

Therefore,

$$u(z, t) = A \exp(-r_1 z \cos\theta_1) \exp\{i(\omega t - r_1 z \sin\theta_1)\}$$
$$+ B \exp(-r_1 z \sin\theta_1) \exp\{-i(\omega t + r_1 z \cos\theta_1)\}$$
$$+ \frac{1}{\pi} \int_0^\infty \left(\frac{A}{x + i\omega} + \frac{B}{x - i\omega}\right) e^{-xt} \sin z \sqrt{\frac{x}{\alpha - \beta x}}\, dx. \quad \ldots(9)$$

Equation (9) represents the equation of the transverse wave spreading out from the oscillating disk. It can be noticed that the amplitude of the waves decays exponentially with z. It is found that the amplitude of the waves decreases with the increase of coefficient of viscoelasticity β and the forcing frequency of the oscillations ω. Thus the effect of viscoelasticity is to reduce the flow velocities.

3. Skin Friction and Energy Dissipation

The skin friction, defined as the shearing stress per unit area exerted by the fluid on the oscillating disk is given by

$$\tau_\omega = \left\{\alpha \frac{\partial u}{\partial z} + \beta \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial t}\right)\right\}_{z=0} = -\alpha r_1 [A \cos(\omega t + \theta_1) + B \sin(\omega t + \theta_1)]$$
$$- r_1 \beta\omega [A \sin(\omega t + \theta_1) + B \cos(\omega t + \theta_1)]$$
$$+ \frac{1}{\pi} \int_0^\infty \left(\frac{A}{x + i\omega} + \frac{B}{x - i\omega}\right) \sqrt{x(\alpha - \beta x)}\, e^{-xt}\, dx. \quad \ldots(10)$$

Hence the skin friction has a phase lead over the induced velocity ($Ae^{i\omega t} + Be^{-i\omega t}$). It increases with the increasing coefficient of viscoelasticity and decreases with the increase of forcing frequency ω.

The rate of energy dissipation from the oscillating disk is given by the mean value :

$$-\overline{(\tau_\omega u)}_{z=0} = (A + B)\, r_1 \Big[\alpha(A \cos\theta_1 - B \sin\theta_1)$$
$$+ \beta\omega (B \cos\theta_1 + A \sin\theta_1)$$
$$- \frac{1}{\pi} \int_0^\infty \left(\frac{A}{x + i\omega} + \frac{B}{x - i\omega}\right) \sqrt{x(\alpha - \beta x)}\, dx\Big]. \quad \ldots(11)$$

Evidently, the effect of elasticity is to increase the rate of energy dissipation from the oscillating disk.

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