PHYSICAL BEHAVIOUR OF SOME CYLINDRICALLY SYMMETRIC BRANS-DICKE FIELDS

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(*Received* 3 *October* 1977)

In this paper we have discussed certain physical aspects, particularly with regard to singularity, of the solutions obtained in our earlier work for cylindrically symmetric Brans-Dicke fields (Rao *et al.* 1974a, b; 1975).

1. INTRODUCTION

The present paper is devoted to the physical interpretation of some of the solutions of our papers entitled 'Cylindrically Symmetric Brans-Dicke Fields I, II, III (hereafterwards referred to as I, II, III respectively)'. We would, however, like to mention at the outset that our interpretation here is mainly with regard to the solutions of I and II only. We further mention that the solutions of I, being the B-D vacuum solutions of Einstein-Rosen metric, are physically interesting in certain aspects. The value of the B-D scalar ϕ (when ϕ is nonstatic), obtained in I, represents the wave character suggesting thereby that the cylindrical scalar waves along with the gravitational waves may exist in the same way as the gravitational waves in the case of the Einstein's theory.

A good part of the work in the physical interpretation here involves the study of the intrinsic singularities of the solutions. Although there is no definite accepted definition of singularities in general relativity as the various authors have proposed different definitions [e.g., completeness of geodesics (Geroch 1968), regularity defined through $g_{\mu\nu}$ and their derivatives (Bonnor 1957)], a common view held by most of them is to study the invariants of the field. These being coordinate independent will reflect the nature of singularity. We have held this view and studied the nature of singularities through some of these invariants, like Riemann curvature invariant, Kretschmann curvature invariant and eigenvalues of the field determined by the energy-momentum tensor $T_{\mu\nu}$.

The investigation carried out here has been divided into two main sections. In Section 2, we have established the validity of the Bonnor's theorem (1954) for constructing the dual solutions of the Einstein-Maxwell field in the case of the B-D Maxwell field. It has been shown through the help of the theorem, that the solutions

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of II are not all independent. In fact, the solutions of the case 3(a) of II are dual to the case 3(b) of II and vice-versa. In the same section we have also studied the regularity conditions due to Bonnor (1957) and have shown that the solutions of I and II do not satisfy these conditions. The Section 3 deals with the study of the singular behaviour of some of the invariants, i.e., Riemann curvature invariant, Kretschmann curvature invariant and the eigenvalues of the field. In these discussions we have considered only two of the solutions, one from I and the other from II. A parallel study can be made for all the other solutions following a similar approach.

2(a). Duality of Solutions

According to Bonnor (1957), if $(g_{\mu\nu}, F_{\mu\nu})$ is a solution of Einstein-Maxwell equations then $(g_{\mu\nu}, F^*_{\mu\nu})$ is also a solution, where

$$F^{*\mu\nu} = \tfrac{1}{2}\eta^{\mu\nu\lambda\delta} F_{\lambda\delta} = -\tfrac{1}{2}\epsilon^{\mu\nu\lambda\delta} (-g)^{-1/2} F_{\lambda\delta} \qquad ...(1)$$

is called the dual of $F_{\mu\nu}$; $\epsilon_{\mu\nu\lambda\delta}$ is the permutation symbol and g the determinant of the metric tensor $g_{\mu\nu}$. The theorem has been found to be true by Roy *et al.* (1973) in the Einstein's gravitational theory for coupled zero mass and electromagnetic fields as well. We have, however, observed here that this theorem holds in the case of B-D Maxwell field also. Thus, if $(g_{\mu\nu}, F_{\mu\nu}, \phi)$ be a solution of the B-D Maxwell field, where ϕ is the B-D scalar field, then $(g_{\mu\nu}, F^{*\mu\nu}, \phi)$ is also a solution with $F^{*\mu\nu}$, the dual, being given by (1). Now the solutions for the cases 3(a.2) and 3(a.3) of III contain the electric components of the electromagnetic tensor $F_{\mu\nu}$, whereas the solutions for the cases 3(b.2) and 3(b.3) contain the magnetic components only. In view of the theorem, therefore, we can always construct their duals. For example, we consider the solution* [II, (25), (34)-(36)], the non-vanishing components $F_{\mu\nu}$ of which are

$$F_{34} = -F_{43} = \frac{q^2}{2u(at+b)} \operatorname{sech}^2 \left[-\frac{q}{2a} \log (at+b) + r \right]. \qquad ...(2)$$

The corresponding dual can be written as

$$F^{*12} = -\tfrac{1}{2}(\epsilon^{1234} F_{34} + \epsilon^{1243} F_{43}) (-g)^{-1/2}$$
$$= -F_{34} (-g)^{-1/2}.$$

Substituting the value of F_{34} from (2), we obtain

$$F^{*12} = f\rho$$

which implies, in view of [II, (14)],

$$\xi = \tfrac{1}{2} f\rho^2 + p. \qquad ...(3)$$

*Hereafterwards the numerals I, II, III refer to Parts I, II and III respectively of the paper entitled 'Cylindrically Symmetric Brans-Dicke Fields, I, II, III' and the numbers given in the parenthesis refer to the equations of the corresponding papers.

Hence, another solution is given by

$$\phi = at + b,$$

$$\xi = \tfrac{1}{2} f \rho^2 + p,$$

$$\beta = \tfrac{1}{2} \log \left[\frac{q^2}{2uf(at+b)} \operatorname{sech}^2 \left(-\frac{q}{2} \log (at+b) + r\right)\right],$$

$$\alpha = \tfrac{1}{4}\left[\tfrac{1}{2}\left(1 - \frac{q^2}{a^2}\right) - \omega\right] \log \left[\frac{1}{\rho^2} - \frac{a^2}{(at+b)^2}\right]$$

$$+ \tfrac{1}{2}\left[\tfrac{1}{2}\left(1 - \frac{q^2}{a^2}\right) - \omega\right] \log \rho + s \qquad \text{...(4)}$$

which is the same as solution [II, (64)]. Similarly it can be verified that the duals of all the solutions for the cases 3(a.2) or (3a.3) for which $\xi = 0$, are actually the solutions already obtained for the cases 3(b.3) or (3b.2) when $\eta = 0$, and vice-versa. In fact, knowing the solution of the first set we can derive the solution of the other set by using the theorem. This suggests that the solutions of II are not all independent.

2(b). Regularity of the Solutions

According to Bonnor (1957) a non-singular point is that at which it is possible to introduce natural coordinates by a coordinate transformation. At a non-singular point, therefore, one should be able to introduce a local tangent plane which is Minkowiskian. A field is said to be non-singular if every point of this (including the points at spatial and temporal infinity) is non-singular.

The sufficient condition for the introduction of natural coordinates at P are

(*i*) g, the determinant of $g_{\mu\nu}$, is nonzero ;

(*ii*) $g_{\mu\nu}$ and their first derivatives are finite and continuous at P ;

(*iii*) the second derivatives of $g_{\mu\nu}$ are finite and continuous at P.

The third condition ensures the finiteness and continuity of the Riemann-Christoffel tensor. The coordinate singularity that may be present is avoided by transforming the Einstein-Rosen metric to pseudo-Cartesian coordinates by the transformation

$$x = \rho \cos \Phi, y = \rho \sin \Phi, z = z, t = t. \qquad \text{...(6)}$$

By this transformation the Einstein-Rosen metric [I, (4)] becomes

$$ds^2 = - e^{2\beta} dz^2 - \rho^{-2} [(x^2 e^{2\alpha-2\beta} + y^2 e^{-2\beta}) dx^2 + (y^2 e^{2\alpha-2\beta} + x^2 e^{-2\beta}) dy^2]$$

$$+ 2xy(e^{-2\beta} - e^{2\alpha-2\beta}) dx\, dy + e^{2\alpha-2\beta} dt^2. \qquad \text{...(7)}$$

Now it can be checked that the solutions [I, (12)], [I, (37)], [II, (25), (34), (35), (36)] and [II, 79] do not satisfy even the first two regularity conditions of (5). Hence these solutions are not everywhere regular. In fact, all of them are singular both along the z-axis and at infinity. Similar study may be carried out for other solutions.

2(c). Nature of the B-D Scalar Field

Here we would like to discuss the singular behaviour of B-D scalar field ϕ which is also an invariant of the field. One can easily verify that for two solutions [I, (12)] and [I, (14), (10), (19)], the scalar ϕ is singular on the axis of z (when $\rho \to 0$). This shows that the source of these two vacuum cylindricaly symmetric B-D scalar fields lies along the axis of z. In the remaining solutions of I and II, ϕ, being a linear function of time only, does not exhibit any singular behaviour. It, however, increases with time which implies that G — the gravitational variable — (gravitation constant is no longer a constant in the B-D theory) decreases with time. Since the decrease of G with time is connected with the decrease of the luminosity of the stars the solutions seem to exhibit the character of some astrophysical system or systems (Dicke 1964).

3. SINGULAR BEHAVIOUR OF INVARIANTS

We now study the regularity of the solutions from the behaviour of some of the invariants, viz., the Kretschmann curvature invariant $\mathcal{L}$, the Riemann curvature invariant R, and the eigenvalues of the field determined by the energy-momentum tensor. A general behaviour of non-null eigenvalues with respect to scalar fields (zero-mass and massive scalar fields of the Einstein's gravitational theory and scalar field of the B-D theory) has been discussed in a separate paper (Tiwari and Bhamra 1977). Throughout the discussions here we have considered only two of the solutions, viz., [I, (37)] and [II, (25), (34)–(36)]. The discussion is divided into four subsections as follows.

3(a). Singular Behaviour of the Curvature Invariant

To determine the behaviour of the curvature invariant R, we first consider the case of the B-D Maxwell field. The curvature invariant in terms of the metric coefficients α and β for the Einstein-Rosen metric is given by

$$R = 2e^{2\beta-2\alpha}\left[-\alpha_{11}+\alpha_{44}+\beta_{11}-\beta_{44}+\beta_4^2-\beta_1^2+\frac{\beta_1}{\rho}\right]. \qquad \text{...(8)}$$

Substituting the values of α and β in [I, (5)], we have from [II, (25), (34) – (36)], the non-vanishing components of the Einstein-tensor G_ν^μ as

$$G_1^1 = -G_4^4,\ G_2^2 = -G_3^3,\ G_4^1 = -G_1^4. \qquad \text{...(9)}$$

The corresponding curvature invariant is given by

$$R = \frac{ma^2\omega\rho^{(\omega-(2u'/m))}}{\left[\left(\frac{at+b}{\rho}\right)^2-a^2\right]^{(1-2\omega)/4}\left[\left(\frac{at+b}{\rho}\right)+a\right]^{2u'/m}(at+b)^{\omega+2}}. \qquad \text{...(10)}$$

Now for t = constant hypersurface, we have

(*i*) $R \to 0$ as $\rho \to 0$

and

(*ii*) $\rho \to \infty$, $R \to \infty$, finite value, or zero according as $\omega - \frac{2u'}{m} \gtreqless 0$.

Similarly keeping ρ constant, we get $t \to \pm\infty$, $R \to \pm\infty$ when $\frac{5}{2} + \frac{2u'}{m} < 0$. Thus at the origin the curvature invariant is always regular whereas at spatial infinity it is finite (i.e. regular) only when $\omega - \frac{2u'}{m} \leqslant 0$. It may further be noted that when ρ is constant and $\frac{5}{2} + \frac{2u'}{m} < 0$, the scalar field and the curvature invariant are singular at every point in the infinite past as well as at temporal infinity.

In the case of vacuum solution [I, (37)] the curvature invariant R is given by

$$R = \frac{2k(k+a)}{m^2}(at+b)^{2(k-a)/a}.$$

The value of R here is independent of space coordinates. For t = constant hypersurface, the expression for R is always finite except for the case when $t = -\frac{b}{a}$ and $k < a$. In this case, however, the B-D scalar field ϕ becomes zero which is impossible from the very formulation of the theory. At temporal infinity the behaviour is different. It can be verified that for $t = \pm\infty$,

(*i*) $R \to 0$ when $k > a$

and

(*ii*) $R \to \infty$ when $k > a$.

3(b). Eigenvalue Behaviour of the Fields

In a separate paper we have obtained (Tiwari and Bhamra 1977) that in the case of B-D vacuum fields when the scalar is a function of all the four coordinates (x^1, x^2, x^3, x^4), the eigenvalues are given by a single biquadratic relation and may, in general, be different from one another. However, for some of the solutions obtained in I and for all the solutions of II, we observe that the scalar ϕ is only a linear function of time. For the B-D Maxwell solution [II, (25), (34)–(36)], therefore, we obtain the eigenvalues directly by simplifying the determinantal equation

$$| T^{\mu}_{\nu} - \lambda\delta^{\mu}_{\nu} | = 0.$$

We define the right hand side of the B-D field equations to be equivalent to some energy momentum tensor $\bar{T}^{\mu}_{\nu}$ (which is equal to G^{μ}_{ν}—the Einstein's tensor). The non-vanishing components of G^{μ}_{ν} are given by the relation (9). Thus the above determinantal equation reduces to

$$(\bar{T}^2_2 - \lambda)(\bar{T}^3_3 - \lambda)[(\bar{T}^1_1 - \lambda)(\bar{T}^4_4 - \lambda) - \bar{T}^1_4 \bar{T}^4_1] = 0.$$

The two eigenvalues λ_2 and λ_3, being given by $\bar{T}^2_2$ and $\bar{T}^3_3$, respectively, are equal in view of (9). The other two eigenvalues are given by the quadratic relation

$$\lambda^2 - \lambda(\bar{T}^1_1 + \bar{T}^4_4) + \bar{T}^1_1 \bar{T}^4_4 - \bar{T}^1_4 \bar{T}^4_1 = 0$$

which in view of (9) reduces to

$$\lambda = \pm [\bar{T}^1_1 \bar{T}^4_4 - \bar{T}^4_1 \bar{T}^1_4]^{1/2}.$$

For the solution [II, (25), (34) – (36)] the corresponding eigenvalues, after calculating, are given by

$$\lambda_2 = \lambda_3 = \frac{ma^2\omega\rho^{1/2}}{\left\{1 - \left(\frac{\rho a}{at+b}\right)^2\right\}^{(1-2\omega)/4}\left\{1 + \frac{\rho a}{at+b}\right\}^{2u'/b} \cdot (at+b)^{(5m+4u')/2m}} \qquad \ldots(11)$$

$$\begin{aligned}\lambda_1 = -\lambda_4 &= \frac{m\rho^{-3/2}}{\left\{1 - \left(\frac{\rho a}{at+b}\right)^2\right\}^{(1-2\omega)/4}\left\{1 + \frac{\rho a}{at+b}\right\}^{2u'/m} \cdot (at+b)^{(4u'+m)/2m}} \\ &\times \Bigg[\left(\frac{a}{at+b}\right)^2\Bigg\{\tfrac{1}{2}(\omega - \tfrac{1}{2}) \left(\frac{-\rho^2\frac{a}{at+b}}{\left\{1 - \left(\frac{\rho a}{at+b}\right)^2\right\}}\right)^2 \\ &- \frac{u'}{m}\left(\rho\Big/\left(1 + \frac{\rho a}{at+b}\right)\right)\Bigg\} \\ &- \rho^2\Bigg\{\tfrac{1}{2}(\tfrac{1}{2} - \omega) \cdot \frac{\rho^2\left(\frac{a}{at+b}\right)^3}{\left[1 - \left(\frac{\rho a}{at+b}\right)^2\right]} + \frac{u'}{m} \cdot \frac{-\rho\left(\frac{a}{at+b}\right)^2}{1 + \left(\frac{\rho a}{at+b}\right)} \\ &+ \left(\frac{u'}{m} + \tfrac{1}{4}\right)\frac{a}{at+b}\Bigg\}^2\Bigg]^{1/2} \cdot \end{aligned} \qquad \ldots(12)$$

Now for t = constant hypersurface, we get

$$\text{as } \rho \to 0,\ \lambda_2 = \lambda_3 \to 0, \text{ and } \lambda_1 = -\lambda_4 \to \infty$$

and

$$\text{as } \rho \to \infty,\ \lambda_i \to 0, \text{ finite}, \infty, \text{ according as } \omega - \frac{2u'}{m} \lesseqgtr 0 \ (i = 1, 2, 3, 4).$$

Similarly keeping ρ as constant, we obtain,

$$\text{as } t \to \pm\infty,\ \lambda_i \to 0, \text{ finite}, \infty, \text{ according as } \frac{2u'}{m} + \tfrac{5}{2} \gtreqless 0.$$

At the origin of t = constant hypersurface $\left(\text{except } t = -\dfrac{b}{a}\right)$, two of the eigenvalues are always zero (viz., finite) whereas the other two tend to infinity. Since λ_i are the values corresponding to the principal directions, it shows that the singularities are exhibited in a preferred direction when we approach the origin. The behaviour at spatial infinity, however, depends on the expression $\omega - \dfrac{2u'}{m}$. Indeed, the behaviour at spatial infinity is similar to that of the curvature invariant R. Similarly, at infinite past and future, the singular behaviour is dependent on the expression $\dfrac{2u'}{m} + \tfrac{5}{2}$.

The eigenvalues of the B-D vacuum field, when ϕ is a linear function of time only, are given as

$$\lambda_2 = -\frac{\omega}{2\phi^2} P + \frac{1}{\phi}\ \phi^{2}_{;2},$$

$$\lambda_3 = -\frac{\omega}{2\phi^2} P + \frac{1}{\phi}\ \phi^{3}_{;3},$$

$$\lambda_1 = -\lambda_4 = \left[\frac{\omega^2}{4}\frac{P^2}{\phi^4} - \frac{\omega}{\phi^3}\phi^4\phi_4 \cdot \phi^{1}_{;1} - \frac{1}{\phi}\{\phi^{1}_{;1}\ \phi^{4}_{;4} - \phi^{1}_{;4}\ \phi^{4}_{;1}\}\right]^{1/2} \qquad \ldots(13)$$

where

$P = \phi^i \phi_i$. The relations (13), in the case of the solution [I, (37)], give

$$\lambda_1 = \lambda_2 = -\lambda_4 = -\frac{a}{m^2}\left(\frac{\omega a}{2} + k\right)(at + b)^{(k-a)/a}$$

$$\lambda_3 = -\frac{a}{m^2}\left(\frac{\omega a}{2} - k\right)(at + b)^{(k-a)/a}. \qquad \ldots(14)$$

For t = constant hypersurface all the eigenvalues are finite and regular when $k > a$. They become singular, however, when $k < a$ and $t = -\frac{b}{a}$ (constant). This is not possible since ϕ becomes zero in this case. Thus, the singular behaviour of the eigenvalues in this case is similar to that of R.

3(c). Study of the Kretschmann Curvature Invariant

In this section we analyse the singular behaviour of the solutions through the nature of Kretschmann curvature invariant given by

$$\mathcal{L} = R_{\mu\nu\lambda\delta}\, R^{\mu\nu\lambda\delta} \qquad \ldots(15)$$

where $R_{\mu\nu\lambda\delta}$ is the well-known Riemann curvature tensor.

The value of $\mathcal{L}$ for the solution [II, (25), (34)–(36)] is obtained, after a straightforward but tedious calculation, as

$$\mathcal{L} = \frac{m^2 \rho^{-3}}{\left\{1 - \left(\frac{\rho a}{at+b}\right)^2\right\}^{(1-4\omega)/4} \left\{1 + \frac{\rho a}{at+b}\right\}^{4u'/m} (at+b)^{(4u'+m)/m}}$$
$$\times \left\{ \tfrac{1}{2}\left(\tfrac{1}{2} - \omega\right) \frac{-\rho\left(\frac{a}{at+b}\right)^2}{1 - \rho^2\left(\frac{a}{at+b}\right)^2} + \frac{u'}{m} \frac{\left(\frac{\rho a}{at+b}\right)}{1 + \left(\frac{\rho a}{at+b}\right)} \right\}^2$$
$$+ \frac{1}{16}(\rho^4 + 1) + \tfrac{1}{4}\left\{\tfrac{1}{2} + \omega\rho^2\left(\frac{a}{at+b}\right)^2\right\}^2$$
$$- \frac{a^2\rho^2}{2(at+b)^2}\left\{\frac{u'}{m}\,\frac{\rho}{1 + \frac{\rho a}{at+b}} + \frac{1}{4\left[1 - \rho^2\left(\frac{a}{at+b}\right)^2\right]}\right.$$
$$\left.- \frac{\omega}{2}\,\frac{\left(\frac{\rho}{at+b}\right)^2}{1 - \left(\frac{\rho a}{at+b}\right)^2}\right\}^2\Bigg]. \qquad \ldots(16)$$

For t = constant hypersurface, we get

as $\rho \to 0$, $\mathcal{L} \to \infty$

and

as $\rho \to \infty$, $\mathcal{L} \to 0$, finite, ∞, according as $\omega - \frac{2u'}{m} \lesseqgtr 0$.

Similarly keeping ρ constant, we have

as $t \to \infty$, $\mathcal{L} \to 0$, finite, ∞, according as $\frac{4u'}{m} + 1 \gtrless 0$.

Thus, for the solution [II, (25), (34) - (36)], the Kretschmann curvature invariant $\mathcal{L}$ is always singular at the origin.

The singular behaviour at spatial infinity is similar to that of the curvature invariant R or the eigenvalues λ_i. At temporal infinity it is dependent on the expression $\frac{4u'}{m} + 1$.

Similarly for the solution [I, (37)] the value of $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{m^4}(7k^4 + 4k^2a^2 - 4k^3a)(at + b)^{4(k-a)/a}. \quad ...(17)$$

In this case also the behaviour of $\mathcal{L}$ is similar to that of the curvature invariant R or the eigenvalues λ_i.

3(d). Conclusions

In the case of the vacuum solution [I, (37)] the behaviour of the three invariants, viz., R, $\mathcal{L}$, λ_i, is similar. In fact, at spatial and temporal infinity they all tend either to zero or infinity according as $k < a$ or $k > a$. We may further mention that for this particular solution the invariants, R, $\mathcal{L}$, and λ_i are independent of space coordinates.

The singular behaviour of these invariants for the B-D Maxwell field solution [II, (25), (34) - (36)] shows an interesting feature at the origin. It has been found that at the origin (i.e., when $\rho \to 0$);

(*i*) the curvature invariant R always tends to zero,

(*ii*) two of the eigenvalues are finite and the remaining two are always infinite,

(*iii*) the Kretschmann curvature invariant $\mathcal{L}$ always tends to infinity.

It may be mentioned that whereas the finiteness of the invariants at any point does not ensure the regularity of the solution, the singularity of even one of them is enough to conclude that the solution is singular at the point under consideration. The origin, therefore, is certainly a singular point in this case. The singularity at the origin which is not at all sensed through R, is permanently exhibited in $\mathcal{L}$. And in λ_i it is exhibited in certain preferred directions (principal direction) only. The relevant directions may be known by calculating the corresponding eigenvectors. We may call such singularities to be the directional singularities. We may mention finally that both, at spatial and temporal infinity, the singular behaviour of the invariants, viz., R, $\mathcal{L}$ and λ_i, is similar in this case. At spatial infinity they all depend on the sign of the expression $\omega - \frac{2u'}{m}$.

ACKNOWLEDGEMENT

The authors thank Prof. G. Bandyopadhyay for his continuous interest in the work.

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