LONGITUDINAL DISPERSION IN SATURATED POROUS MEDIA

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Coefficient of longitudinal dispersion has been discussed for steady uniform laminar flow through saturated porous media. Laplace transform technique has been applied to solve the equation for concentration. Results have been tabulated and some variations have been illustrated by graphs. It has been concluded that the concentration decreases and also is in decreasing order as distance increases parallel to the direction of flow. More interesting results have been discussed.

1. INTRODUCTION

It is known that the mass-transport perpendicular to an unidirectional flow through porous media is larger than mass-transport in the direction of flow. For steady uniform flow parallel to the x-axis in saturated homogeneous porous media, the equation for conservation of mass of the displacing substance can be written as (Scheidegger 1961, Harleman and Rumer 1962).

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D \frac{\partial^2 C}{\partial x^2} \qquad \ldots(1)$$

where, C is the concentration of dispersion, u the average velocity of the fluid, D the coefficient of longitudinal dispersion, x the coordinate parallel to the flow and t the time.

Scheidegger (1961) postulated that the coefficient of dispersion is related to the pore system geometry and the seepage velocity by the expression

$$D = \gamma \mid u \mid$$

where γ is the coefficient having the dimension of light and dependent upon the pore system geometry. The absolute value of the seepage velocity is used, because negative dispersion coefficient is meaningless. Some investigators (Raimandi *et al.* 1959, Mehlhorn 1962) have found that γ is proportional to the average grain size of a porous medium, i.e.

$$\gamma = \rho d$$

where ρ is a coefficient of proportionality dependent upon the particle shape and size distribution and d the average grain size.

The mixing of miscible fluids for flow through porous media is generally referred to as dispersion. Interest in dispersion in porous media has resulted from water quality considerations of artificial rechange and waste water disposal operations (see water intrusion into coastal aquifers and seepage from canals and streams into and through aquifers).

Many investigators (Bruch and Street 1966, Harleman and Rumer 1963, Hoopes and Harleman 1965, Ogata and Banks 1961, Shamir and Harleman 1961) have obtained solutions to dispersion problems in porous media. Common to these studies is the assumption of a set function from input concentration, i.e. the input concentration is changed instantaneously from zero to some value and is maintained at this value thereafter.

This paper describes mathematical solutions to two simplified field dispersion problems involving variable input concentration of contaminants. Marino (1974) has also studied the problem of longitudinal dispersion in saturated porous media and developed a good approximation to the solution of the problem.

We have taken the same problem as considered by Marino (1974). In this paper boundary conditions have been changed. Calculations have been done to take only real part of the boundary condition, which gives the different results from that of Marino.

2. Theory

The theory that follows is limited to unsteady longitudinal dispersion in idealized steady one dimensional seepage flows through semi-infinite homogeneous isotropic porous media. The solutions to be developed predict the concentration of contaminants as a function of time and space, if in seepage flow the longitudinal dispersion coefficient and the source concentration are prescribed. Adsorption is neglected, only simple dispersion systems are treated.

The equation for longitudinal hydrodynamic dispersion within semi-infinite medium in a unidirectional flow field has been given by eqn. (1).

Case I

The system under consideration is schematically represented in Fig. 1. The concentration of the displacing fluid at $x = 0$ is $C_0 e^{i\omega t}$ where $\omega =$ constant, positive or negative.

We can write the longitudinal dispersion problem as eqn. (1) with initial and boundary conditions :

$$C(x, 0) = 0, \qquad x \geqslant 0 \qquad \ldots(2)$$

$$C(0, t) = C_0 e^{i\omega t}, \; t > 0 \qquad \ldots(3)$$

$$C(\infty, t) = 0, \qquad t \geqslant 0. \qquad \ldots(4)$$

It is convenient to transform eqn. (1) so that the convective term does not appear explicitly, using the transformation (Ogata and Banks 1961)

$$C(x, t) = C^*(x, t) \exp i\left(\frac{ux}{2D} - \frac{u^2t}{4D}\right) \qquad \ldots(5)$$

and considering the corresponding real part of condition (3), eqns. (1) – (4) become

$$\frac{\partial^2 C^*}{\partial x^2} = \frac{1}{D}\frac{\partial C^*}{\partial t} \qquad \ldots(6)$$

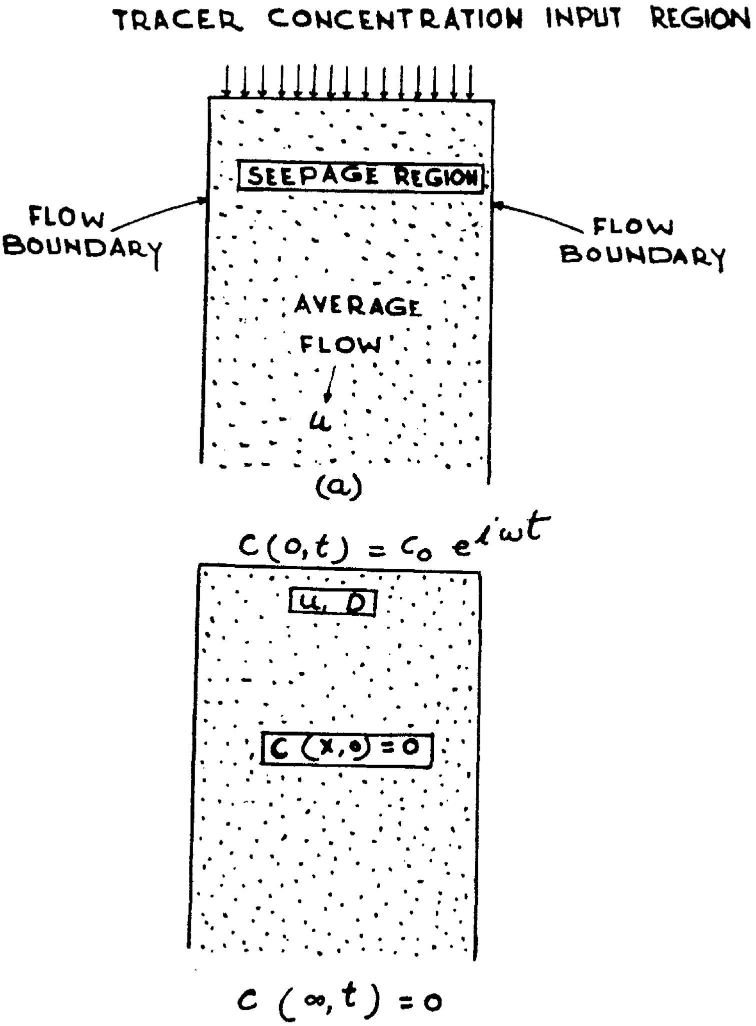

FIG. 1. Semi-infinite porous medium in unidirectional flow field, concentration of displacing fluid is $C_0e^{i\omega t}$ in the fluid.

$$C^*(x, 0) = 0 \qquad \text{...(7)}$$

$$C^*(0, t) = C_0 \cos \eta t \qquad \text{...(8)}$$

$$C^*(\infty, t) = 0 \qquad \text{...(9)}$$

where

$$\eta = \frac{u^2}{4D} + \omega. \qquad \text{...(10)}$$

Let $\bar{C}^*(x, p)$ be the transform, with respect to t of the concentration function $C^*(x, t)$. Applying the Laplace transformation with respect to t on the preceding boundary value problem using eqn. (7), we have

$$\frac{\partial^2 \bar{C}^*}{\partial x^2} - \frac{p}{D} \bar{C}^* = 0 \qquad \text{...(11)}$$

$$\bar{C}^*(0, p) = C_0 \frac{p}{p^2 + \eta^2} \qquad \text{...(12)}$$

$$\bar{C}^*(\infty, p) = 0 \qquad \text{...(13)}$$

where p is the parameter of the transformation. The general solution of the ordinary differential eqn. (11) is

$$\bar{C}^*(x, p) = a_1 \exp [- x(p/D)^{1/2}] + a_2 \exp [x(p/D)^{1/2}] \qquad \text{...(14)}$$

where a_1 and a_2 are constants of integration.

Using eqns. (12), (13) in eqn. (14), the constants are determined as

$$a_1 = C_0 p/p^2 + \eta^2 \qquad \text{...(15)}$$

$$a_2 = 0 \qquad \text{...(16)}$$

and

$$\bar{C}^*(x, p) = C_0 \frac{p}{p^2 + \eta^2} \exp [- x(p/D)^{1/2}]. \qquad \text{...(17)}$$

Laplace table (Churchill 1958) gives

$$\frac{p}{p^2 + \eta^2} = L(\cos \eta t) \qquad \text{...(18)}$$

and

$$\exp (- x \sqrt{p/D}) = L \left\{ \frac{x}{2 \sqrt{\pi D t^3}} \exp (- x^2/4Dt) \right\}. \qquad \text{...(19)}$$

We can write, with the aid of the convolution for each fixed θ ($\theta \geqslant 0$)

$$C^*(x, t) = C_0 \int_0^t \cos \eta(t-\theta) \frac{x}{2\sqrt{\pi D\theta^3}} \exp(-x^2/4D\theta)\, d\theta. \qquad \text{...(20)}$$

Substituting a new variable of integration

$$\lambda = x/2\sqrt{D\theta}$$

we have

$$C^*(x, t) = C_0 \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{Dt}}^{\infty} \cos \eta \left(t - \frac{x^2}{4D\lambda^2}\right) e^{-\lambda^2}\, d\lambda. \qquad \text{...(21)}$$

This expression shows that if the surface concentration in the semi-infinite porous medium, $x = 0$, is given by $C^* = C_0 \cos \eta t$, we have then the initial concentration as zero.

Now it is known that

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-\lambda^2} \cos\left(\frac{\omega x^2}{4D\lambda^2} + \phi\right) d\lambda$$

$$= \exp(-x\sqrt{\omega/2D}) \cos\left\{x\left(\frac{\omega}{2D}\right)^{1/2} + \phi\right\}. \qquad \text{...(22)}$$

Therefore, when t is so great that the lower limit of the integral in (21) may be taken to be zero, then

$$C^*(x, t) = C_0 \exp(-x\sqrt{\eta/2D}) \cos\left\{\eta t - x\left(\frac{\eta}{2D}\right)^{1/2}\right\}. \qquad \text{...(23)}$$

Now

$$C(x, t) = C_0 \exp(-x\sqrt{(u^2 + 4D\omega)/8D^2}) \cos\left\{\left(\frac{u^2 + 4D\omega}{4D}\right) t - x\left(\frac{u^2 + 4D\omega)}{8D^2}\right)^{1/2}\right\} \cos\left(\frac{ux}{2D} - \frac{u^2 t}{4D}\right) \qquad \text{...(24)}$$

where

$$\eta = \frac{u^2}{4D} + \omega.$$

Note that if $\omega = 0$, the eqn. (24) becomes

$$C(x, t) = C_0 \exp(-xu/2\sqrt{2D}) \cos\left(\frac{ux}{2\sqrt{2D}} - \frac{u^2t}{4D}\right) \times \cos\left(\frac{ux}{2D} - \frac{u^2t}{4D}\right) \quad ...(25)$$

which is the same as derived by Carslaw and Jaeger (1959) for the case in which the displacing fluid has a constant concentration C_0 and x is in the positive direction.

Case II

The porous medium is represented in Fig. 1, except that the concentration of the displacing fluid at $x = 0$ is $C_0(1 - e^{-i\omega t})$. Using the transformation indicated by eqn. (5), and considering the real part of the equation, the hydrodynamic problem is reduced to

$$\frac{\partial^2 C^*}{\partial x^2} = \frac{1}{D}\frac{\partial C^*}{\partial t} \quad ...(26)$$

$$C^*(x, 0) = 0 \quad ...(27)$$

$$C^*(0, t) = C_0[\cos \alpha t - \cos \beta t] \quad ...(28)$$

$$C^*(\infty, t) = 0 \quad ...(29)$$

where

$$\alpha = \frac{u^2}{4D} \quad ...(30)$$

$$\beta = \frac{u^2}{4D} - \omega. \quad ...(31)$$

Now applying the Laplace transformation, we get

$$\frac{\partial^2 \bar{C}^*}{\partial x^2} - p/D\, \bar{C}^* = 0 \quad ...(32)$$

$$\bar{C}^*(0, p) = C_0\left[\frac{p}{p^2 + \alpha^2} - \frac{p}{p^2 + \beta^2}\right] \quad ...(33)$$

$$\bar{C}^*(\infty, p) = 0. \quad ...(34)$$

The general solution of the ordinary differential eqn. (32) is

$$\bar{C}^*(x, p) = b_1 \exp[-x(p/D)^{1/2}] + b_2 \exp[x(p/D)^{1/2}] \quad ...(35)$$

where b_1 and b_2 are constants of integration. With the help of eqns. (33) and (34), we get

$$b_1 = C_0 \left[\frac{p}{p^2 + \alpha^2} - \frac{p}{p^2 + \beta^2} \right] \qquad \text{...(36)}$$

$$b_2 = 0. \qquad \text{...(37)}$$

Therefore eqn. (35) becomes

$$\bar{C}^*(x, p) = C_0 \left[\frac{p}{p^2 + \alpha^2} - \frac{p}{p^2 + \beta^2} \right] \exp\left[-x(p/D)^{1/2}\right]. \qquad \text{...(38)}$$

Solving in the usual manner we ultimately get

$$\begin{aligned} C(x, t) = {} & C_0 \exp\left(-x/2\sqrt{2}\right) \frac{u}{D} \cos\left(\frac{ux}{2\sqrt{2D}} - \frac{u^2 t}{4D}\right) \\ & \times \cos\left(\frac{ux}{2D} - \frac{u^2 t}{4D}\right) - C_0 \exp\left(-x\sqrt{u^2 - 4D\omega}/2\sqrt{2D}\right) \\ & \times \cos\left\{\frac{(u^2 - 4D\omega)^{1/2}}{2\sqrt{2D}}\, x - \left(\frac{u^2 - 4D\omega}{4D}\right) t\right\} \\ & \times \cos\left(\frac{ux}{2D} - \frac{u^2 t}{4D}\right). \end{aligned} \qquad \text{...(39)}$$

This is the case when $\omega \neq 0$, where x is in the positive direction.

Particular Case : *When* $C = 0$, *if* $x = 0$

From eqn. (39), we have

$$\frac{u^2 t}{4D} = \frac{\omega t}{2} + n\pi. \qquad \text{...(40)}$$

For the particular value, when $n = 0$, we have for $t \neq 0$

$$\omega = \frac{u^2}{2D}. \qquad \text{...(41)}$$

3. Results

Solutions of eqn. (25) are presented graphically in Figs. 2, 3 and 4 for values of D and u, representative of those reported by Ogata and Banks (1961) and Shamir and Harleman (1961). For values of $D = 0.6$ cm²/min and $u = 6$ cm/min, the second term of eqn. (39) is negligible. If we neglect the second term of eqn. (39) we see that eqns. (25) and (39) are the same.

4. Conclusion and Discussions

Analytically solutions have been developed for two longitudinal dispersion problems in saturated porous media. The porous media considered is non-adsorbing, homogeneous. The seepage flows are assumed to be unidirectional and the average velocities are taken to be constant throughout the flow fields.

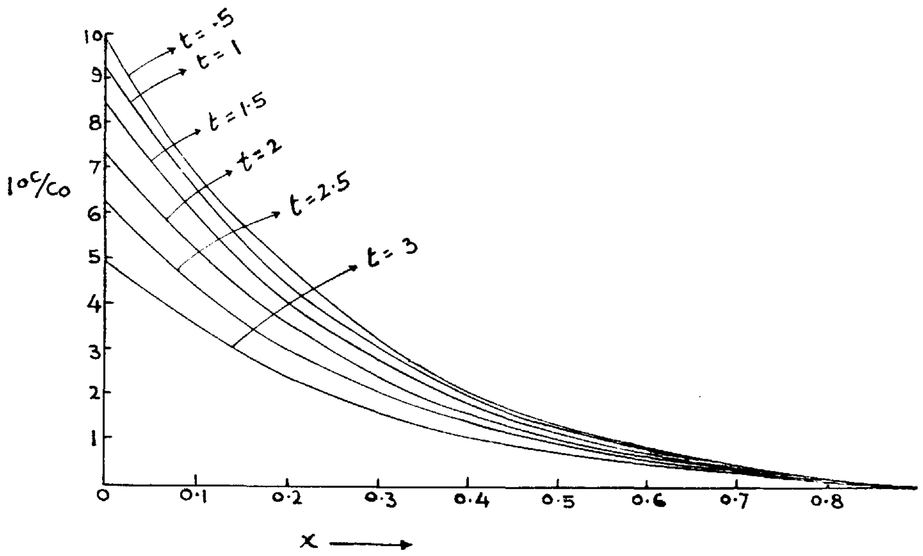

FIG. 2. The variation of concentration against the distance along the flow field for various values of the time. $u = 6$ cm/min, $D = 0.6$ cm^2/min.

FIG. 3. The variation of C/C_0 against the distance. $u = 6$ cm/min, $D = 0.6$ cm^2/min.

Equations (25) and (39) are respective expressions for the concentration, when $\omega = 0$ and $\omega \neq 0$. The numerical values of the concentration are shown in Tables I, II and III. It is clear from Table I that as t increases, the value of concentration decreases and also concentrations are in decreasing order as x increases. From Table II we have similar conclusion as in the Table I. But it is evident from Table II that the value of concentration is approximated towards zero as x increases and at $x \geqslant 1$ it shows no variations for any value of t. In Table III, variations of concentration are shown with respect to t at $x = 0$ and it is clear that as t increases from 0 to 6, the value of concentration decreases from unity to zero which has been shown in Fig. 4. Fig. 2 shows that for increasing t, concentrations are lower. In Fig. 2, the graphs are drawn for concentration versus x for different values of t. It is clear that as t increases graphs are gradually becoming straight. In Fig. 4 the graphs are drawn for concentration versus t and it shows the gradual variations from 0.5 to 6.

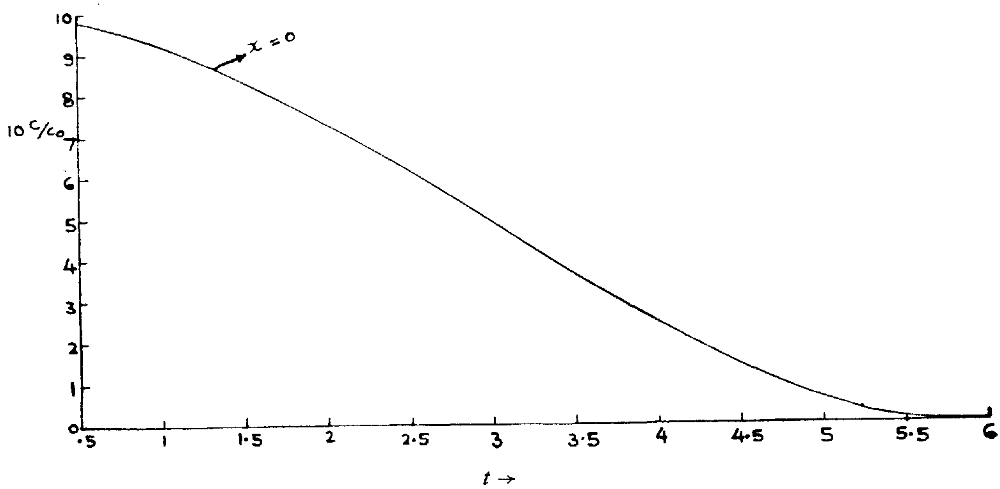

FIG. 4. The variation of the concentration against the time at $x = 0$. $u = 6$ cm/min, $D = 0.6$ cm²/min.

TABLE I

Values of C/C_0 from eqn. (25) *when $u = 6$ cm/min and $D = 0.6$ cm²/min*

x \ t	0.5	1	1.5	2	2.5	3
0	0.98	0.92	0.84	0.73	0.62	0.49
0.1	0.68	0.64	0.59	0.51	0.44	0.35
0.2	0.47	0.44	0.40	0.36	0.30	0.24
0.3	0.32	0.30	0.28	0.24	0.21	0.16
0.4	0.21	0.20	0.18	0.16	0.13	0.11
0.5	0.17	0.16	0.14	0.12	0.10	0.08
0.6	0.09	0.09	0.08	0.07	0.06	0.05
0.7	0.06	0.06	0.06	0.05	0.04	0.03
0.8	0.02	0.02	0.02	0.02	0.01	0.01

TABLE II

Value of C/C_0 when $u = 6.0$ cm/min and $D = 0.6$ cm²/min

x \ t	0.5	1	1.5	2	2.5	3
0	0.98	0.92	0.84	0.73	0.62	0.49
1	0.01	0.01	0.01	0.01	0.01	0.01
2	0.001	0.001	0.001	0.001	0.001	0.001
3	0.00003	0.00003	0.00003	0.00003	0.00003	0.00003
4	0.000001	0.000001	0.000001	0.000001	0.000001	0.000001

TABLE III

Value of C/C_0 for $D = 0.6$ cm²/min and $u = 6.0$ cm/min

x \ t	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
0	1.0	0.98	0.92	0.84	0.73	0.62	0.49	0.36	0.25	0.14	0.06	0.01	0

For particular case of eqn. (39) we see that $C = 0$ at $x = 0$, ω comes out to be 15 when $n = 0$.

The results should prove useful in analyzing seepage of high salt concentration in drainage, ditches, canals and streams.

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