ON THE GEOMETRY OF RELATIVISTIC ELECTROMAGNETIC FLUID FLOWS WITH GEOMETRICAL SYMMETRIES

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In this paper, certain theorems have been established for the relativistic electromagnetic fluid admitting a Ricci collineation and affine collineation with respect to the flow vector, the electric field vector and the magnetic field vector. Conformal motions with respect to the flow vector, the electric field vector and the magnetic field vector are also investigated.

1. Introduction

The search for geometrical symmetries like motions and collineations is motivated by the necessity of discovering conservation laws in the theory of relativity. The special geometrical symmetries which generate "weak" conservation laws are investigated by Davis (1974). Using the method of Davis, we study the weak conservation laws based upon the existence of collineations and motions in self-gravitating universe filled with electromagnetic fluid.

Let us introduce the vectors u^i, a^i, b^i, n^i $(i = 1, 2, 3, 4)$ at a point on the observer's world line such that u^i is a unit time-like vector tangential to the world line and a^i, b^i, n^i are space-like unit vectors satisfying the conditions

$$u^i u_i = 1,\ a^i a_i = b^i b_i = n^i n_i = -1 \qquad \ldots(1.1)$$

$$u^i a_i = u^i b_i = u^i n_i = a^i b_i = a^i n_i = b^i n_i = 0 \qquad \ldots(1.2)$$

in a four dimensional space whose space-time metric is of signature $(+, -, -, -)$.

Let us take u^i as the velocity vector (flow vector) of the fluid, a^i the unit magnetic field vector, b^i the unit electric field vector and n^i the unit electromagnetic energy flux vector. The kinematical quantities (see Appendix for details) associated with congruences generated by u^i, a^i and b^i are given (Prasad and Sinha 1978)

$$u_{i;j} = \sigma_{ij} + w_{ij} + \tfrac{1}{3}\theta\, \gamma_{ij} + (Du_i)\, u_j \qquad \ldots(1.3)$$

$$a_{i;j} = \overset{*}{\sigma}_{ij} + \overset{*}{w}_{ij} + \tfrac{1}{3}\overset{**}{\theta}\, \gamma_{ij} - (D^* a_i)\, a_j \qquad \ldots(1.4)$$

$$b_{i;j} = \overset{\vee}{\sigma}_{ij} + \overset{\vee}{w}_{ij} + \tfrac{1}{3}\overset{\vee\vee}{\theta}\, \gamma_{ij} - (\overset{\vee}{D} b_i)\, b_j \qquad \ldots(1.5)$$

respectively. Here semicolon denotes the covariant differentiation. D, $\overset{*}{D}$ and $\overset{\vee}{D}$ denote the absolute differentiation along the streamlines, the magnetic field lines and the electric field lines respectively.

2. Field Equations

We consider a viscous, compressible, thermally conducting, non-inductive, charged, self-gravitating fluid. We call it a relativistic electromagnetic fluid and the corresponding stress-energy tensor is chosen as (Prasad and Sinha 1978)

$$T^{ij} = \hat{\rho} u^i u^j - \hat{p} g^{ij} - (\lambda e^i e^j + \mu h^i h^j) + \nu \sigma^{ij} + P^i u^j + P^j u^i \qquad \ldots(2.1)$$

where e^i, h^i, P^i and ν ($\geqslant 0$) denote the electric field vector, the magnetic field vector, the energy-flux vector and the coefficient of shear viscosity respectively and

$$\hat{\rho} = \rho + p - \beta\theta + \lambda \mid e \mid^2 + \mu \mid h \mid^2 \qquad \ldots(2.2)$$

$$\hat{p} = p - \beta\theta + \tfrac{1}{2} (\lambda \mid e \mid^2 + \mu \mid h \mid^2) \qquad \ldots(2.3)$$

$$P^i = q^i - v^i,\ e^i e_i = - \mid e \mid^2 ;\ h^i h_i = - \mid h \mid^2 \qquad \ldots(2.4)$$

$$u^i q_i = u^i e_i = u^i h_i = e^i h_i = u^i v_i = e^i v_i = h^i v_i = 0 \qquad \ldots(2.5)$$

where ρ is the matter energy density, p the isotropic pressure, β the coefficient of bulk viscosity, $\frac{1}{2}\lambda \mid e \mid^2$ a supplementary energy density due to electric field, $\frac{1}{2} \mu \mid h \mid^2$ a supplementary energy density due to magnetic field, λ the dielectric capacity and μ the magnetic permeability.

The expression for the heat energy flux vector is given as (Eckart 1940)

$$q^i = - k(T_j - TDu_j)\, \gamma^{ij} \qquad \ldots(2.6)$$

where k is the coefficient of heat conduction and T is the rest temperature.

The field equations for the relativistic electromagnetic fluid are Einstein's equations

$$R_{ij} - \tfrac{1}{2} Rg_{ij} = - T_{ij} \qquad \ldots(2.7)$$

where R_{ij} is usual Ricci tensor. Units are chosen such that $8\pi G$ (gravitational constant) $= c$ (velocity of light) $= 1$.

The relations connecting the thermodynamical variables in the analysis are (Date 1973)

$$TdS = di + pd\left(\frac{1}{\rho_0}\right) \qquad \ldots(2.8)$$

$$\rho = \rho_0(1 + i) \qquad ...(2.9)$$

$$\chi = 1 + i + \frac{p}{\rho_0} \qquad ...(2.10)$$

$$S^i = \rho_0 S u^i + q^i/T \qquad ...(2.11)$$

where ρ_0 is the proper matter density, i is the internal energy density, χ is the fluid index and S^i is the entropy flux vector.

Theorem 2.1 — The entropy flux vector is orthogonal to the magnetic field vector if the magnetic field lines are free from the expansion, the rotation and the shear.

PROOF : Contracting (2.11) with h_i, we get

$$S^i h_i = \frac{1}{T}(q^i h_i) \qquad ...(2.12)$$

Using the fact $q^i h_i = 0 \Leftrightarrow \overset{*}{\theta} = \overset{*}{w}{}^{ij} = \overset{*}{\sigma}{}^{ij} = 0$ (Prasad and Sinha 1978) in (2.12), we conclude the Theorem 2.1.

Theorem 2.2 — The entropy flux vector is orthogonal to the electric field vector iff the electric field lines are free from the expansion, the rotation and the shear.

The proof runs as in Theorem 2.1.

Theorem 2.3 — The rest temperature is constant along the magnetic field lines iff the magnetic field lines are free from the expansion, the rotation and the shear.

PROOF : Transvecting (2.6) with h_i and using the fact

$$q^i h_i = h^i D u_i = 0 \Leftrightarrow \overset{*}{\theta} = \overset{*}{w}{}^{ij} = \overset{*}{\sigma}{}^{ij} = 0$$

(Prasad and Sinha 1978), we get the Theorem 2.3.

Theorem 2.4 — The rest temperature is constant along the electric field lines if the electric field lines are free from the expansion, the rotation and the shear.

The proof runs on the lines of Theorem 2.3.

Theorem 2.5 — The total energy density of the relativistic electromagnetic fluid is constant along the streamlines if the streamlines are free from the expansion, the rotation and the shear.

PROOF : The conservation equations $T^{ij}_{;j} = 0$ yield the local energy balance equation (Date 1973) in the form

$$(D\hat{\rho})\, u^i + \hat{\rho} Du^i + \hat{\rho}\theta u^i - \hat{p}_{,j}\, g^{ij} - \{\lambda_{,j}\, e^i e^j + \lambda\, e^j_{;j}\, e^i$$

$$+ \lambda e^i_{;j}\, e^j + \mu_{,j} h^i h^j + \mu h^i_{;j}\, h^j + \mu h^i\, h^j_{;j}\} + \nu \sigma^{ij}_{;j}$$

$$+ P^i_{;j} u^j + \theta P^i + P^j_{;j} u^i + P^j u^i_{;j} = 0. \qquad \ldots(2.13)$$

Contracting (2.13) with u_i, we get

$$D(\rho + \tfrac{1}{2}\lambda \mid e \mid^2 + \tfrac{1}{2}\mu \mid h \mid^2) + (\rho + p - \beta\theta)\,\theta + (\lambda e^i e^j + \mu h^i h^j)\,\sigma_{ij}$$

$$+ \tfrac{4}{3}\theta(\lambda \mid e \mid^2 + \mu \mid h \mid^2) - 2\nu\,\sigma^2 - P^i Du_i + P^i_{;i} = 0. \qquad \ldots(2.14)$$

Using the fact $P^i = 0 \Leftrightarrow \theta = w^{ij} = \sigma^{ij} = 0$ (Prasad and Sinha 1978) in (2.14), we get Theorem 2.5.

3. Ricci Curvature Collineation

In this section, we deal with the space-time of relativistic electromagnetic fluid admitting the Ricci collineation w.r.t. the flow vector, the magnetic field vector and the electric field vector.

Theorem 3.1 — For a relativistic electromagnetic fluid admitting a Racci collineation w.r.t. the fluid flow vector, (I) the streamlines are expansion free iff the energy-flux vector P^i is divergence free ; and (II) the streamlines are geodesics iff the energy-flux vector P^i remains invariant along the system of the streamlines.

Proof : Combining (2.1) and (2.7), we get

$$R_{ij} = Au_i u_j + Bg_{ij} + \lambda e_i e_j + \mu h_i h_j - \nu\sigma_{ij} - P_i u_j - P_j u_i \qquad \ldots(3.1)$$

where

$$A = -(\rho + p - \beta\theta + \lambda \mid e \mid^2 + \mu \mid h \mid^2) \qquad \ldots(3.2)$$

$$B = \tfrac{1}{2}(\rho - p + \beta\theta + \lambda \mid e \mid^2 + \mu \mid h \mid^2) \qquad \ldots(3.3)$$

we take

$$\mathcal{L}_u R_{ij} = 0 \qquad \ldots(3.4)$$

where $\mathcal{L}_u$ denotes the Lie-derivative w.r.t. u^i.

Equations (3.1) and (3.4) yield

$$A_{,k}u^k u_i u_j + 2ADu_{(i}u_{j)} + B_{,k}u^k g_{ij} + \lambda_{,k}u^k e_i e_j$$

$$+ \mu_{,k}u^k h_i h_j + 2\lambda u^k e_{[k;i]} e_j + 2Bu_{(i;j)}$$

$$+ 2\lambda u^k e_{[j;k]} e_i + 2\mu u^k h_{[i;k]} h_j + 2\mu u^k h_{[j;k]} h_i$$

$$- \nu\{\sigma_{ij;k}u^k + 2u^k{}_{;(i}\sigma_{j)k}\} - 2DP_{(i}u_{j)}$$

$$- 2P_{(i}Du_{j)} - 2u_{k;(i}u_{j)}\, P^k = 0 \qquad \ldots(3.5)$$

where comma denotes partial differentiation. The round bracket and square bracket indicate symmetrization and anti-symmetrization respectively.

Contracting (3.5) with $u^i u^j$, we have

$$(A + B)_{,k}\, u^k = 0. \qquad \ldots(3.6)$$

The conservation law generator (Collinson 1970)

$$(R^i_j\, u^j)_{;i} = 0$$

gives

$$(A + B)_{,k}\, u^k + (A + B)\theta - P^k_{;k} = 0. \qquad \ldots(3.7)$$

By virtue of (3.6) and (3.7), we get

$$(A + B)\,\theta - P^i_{;i} = 0. \qquad \ldots(3.8)$$

Since $A + B \neq 0$, (3.8) proves the statement (I).

Using (3.6) in (3.5) and transvecting with u^j, we obtain

$$(A + B)\, Du_i - (P_{i;k} - P_{k;i})\, u^k = 0. \qquad \ldots(3.9)$$

If the energy-flux vector P_i is invariant along the system of streamlines, we have

$$\mathcal{L}_u P_i = 0 \Rightarrow (P_{i;k} - P_{k;i})\, u^k = 0 \qquad \ldots(3.10)$$

Since $A + B \neq 0$, the statement (II) follows atonce from (3.9) and (3.10).

Corollary 3.1 — If the relativistic electromagnetic fluid with $|P|_{,i}\, P^i = 0$ admits a Ricci collineation w.r.t. the geodetic streamlines, then

$$\theta = 0 \Leftrightarrow \underset{\vee}{\theta} = 0$$

where θ is the expansion parameter of the space-like congruence generated by P^i and

$$|P| = (-P_i P^i)^{1/2}.$$

PROOF : We have the expression for the expansion parameter (Greenberg 1970) of P^i as

$$\underset{\vee}{\theta} = \tfrac{1}{2}\left\{\left(\frac{P^i}{|P|}\right)_{;i} - \left(\frac{P_i}{|P|}\right)_{;j} u^i u^j\right\}$$

for $Du^i = 0$ and $|P|_{,i} P^i = 0$, the above expression reduces to

$$\underset{\vee}{\theta} = \tfrac{1}{2}\left(\frac{P^i_{;i}}{|P|}\right). \qquad ...(3.11)$$

In view of the Theorem 3.1, the Corollary 3.1 follows from (3.11).

Theorem 3.2 — When the streamlines and the electric field lines are free from the expansion, the rotation and the shear, the space-time of the relativistic electromagnetic fluid admits a Ricci collineation w.r.t. the electric field vector if $(\rho + 3p + \lambda |e|^2 + \mu |h|^2)$ is constant along the electric field lines.

PROOF : Following the pattern of the proof of Theorem 3.1 and using the results (Prasad and Sinha 1978)

$$P^i = 0 \Leftrightarrow \theta = w^{ij} = \sigma^{ij} = 0;$$

$$\underset{\vee}{\theta} = \underset{\vee}{\sigma}_{ij} = \underset{\vee}{w}_{ij} = 0 \Leftrightarrow e^i Du_i = 0 \Rightarrow q^i e_i = 0$$

we get Theorem 3.2.

Theorem 3.3 — When the streamlines and the magnetic field lines are free from the expansion, the rotation and the shear, the space-time of the relativistic electromagnetic fluid admits a Ricci collineation w.r.t. the magnetic field vector if $(\rho + 3p + \lambda |e|^2 + \mu |h|^2)$ is constant along the magnetic field lines.

The proof runs on the lines of Theorem 3.2.

4. AFFINE COLLINEATION

An affine collineation w.r.t. the arbitrary vector ξ^i is defined as (Katzin *et al.* 1969)

$$\mathcal{L}_\xi \Gamma^i_{jk} = \xi^i_{;kj} + R^i_{klj}\,\xi^l = 0 \qquad ...(4.1)$$

where Γ^i_{jk} is the usual Christoffel symbol and R^i_{klj} is the Riemann curvature tensor.

Theorem 4.1 — The gradient of the total energy density along the flow is balanced by the energy flux vector when the electric and magnetic field lines are geodesics and the relativistic electromagnetic fluid admits an affine collineation w.r.t. the flow vector u^i which generates the expansion free geodetic streamlines.

PROOF : From (4.1), we have

$$u^i_{;jk} + R^i_{jlk}\,u^l = 0. \qquad ...(4.2)$$

Contracting (4.2) with u_i and using (1.1) and (1.3), we get

$$2w^2 + 2\sigma^2 + \tfrac{1}{3}\theta^2 + Du^i Du_i - R_{ij} u^i u^j = 0 \qquad \ldots(4.3)$$

where

$$2w^2 = w_{ij}w^{ij};\ 2\sigma^2 = \sigma_{ij}\sigma^{ij}.$$

Combining (4.3) with Raychaudhuri's (1955) equation

$$D\theta + \tfrac{1}{3}\theta^2 - (Du^i)_{;i} + 2(\sigma^2 - w^2) + R_{ij} u^i u^j = 0 \qquad \ldots(4.4)$$

we get

$$D\theta + \tfrac{2}{3}\theta^2 + 4\sigma^2 - (Du^i)_{;i} + (Du^i)(Du_i) = 0. \qquad \ldots(4.5)$$

By virtue of (4.5) and (2.14), we obtain

$$\begin{aligned} &D(\rho + \tfrac{1}{2}\lambda \mid e \mid^2 + \tfrac{1}{2}\mu \mid h \mid^2) + (\rho + p - \beta\theta)\,\theta \\ &+ (\mu \mid h \mid^2 u_{i;j}\, a^i a^j + \lambda \mid e \mid^2 u_{i;j}\, b^i b^j) \\ &+ \tfrac{5}{3}\theta(\lambda \mid e \mid^2 + \mu \mid h \mid^2) - \frac{\nu}{2}\{(Du^i)_{;i} - (Du^i)(Du_i) \\ &- D\theta - \tfrac{2}{3}\theta^2\} - P^i Du_i + P^i_{;i} = 0. \end{aligned} \qquad \ldots(4.6)$$

Using (1.1), (1.2), (1.4) and (1.5) in (4.6), we get

$$\begin{aligned} &D(\rho + \tfrac{1}{2}\lambda \mid e \mid^2 + \tfrac{1}{2}\mu \mid h \mid^2) + (\rho + p - \beta\theta)\,\theta - (\mu \mid h \mid^2 u_i \overset{*}{D}a^i \\ &+ \lambda \mid e \mid^2 u_i \overset{\vee}{D}b^i) + \tfrac{5}{3}\theta(\lambda \mid e \mid^2 + \mu \mid h \mid^2) - \tfrac{1}{2}\nu\{(Du^i)_{;i} \\ &- (Du^i)(Du_i) - D\theta - \tfrac{2}{3}\theta^2\} - P^i Du_i + P^i_{;i} = 0. \end{aligned} \qquad \ldots(4.7)$$

On the assumption that the electric and magnetic field lines are geodesics, i.e. $\overset{*}{D}a^i = \overset{\vee}{D}b^i = 0$, and the streamlines are the expansion free and geodesics, i.e. $Du^i = \theta = 0$, we get from (4.7)

$$D(\rho + \tfrac{1}{2}\lambda \mid e \mid^2 + \tfrac{1}{2}\mu \mid h \mid^2) = - P^i_{;i} \qquad \ldots(4.8)$$

which proves the statement.

Theorem 4.2 — If the relativistic electromagnetic fluid admits an affine collineation w.r.t. the unit magnetic field vector a^i and the unit electric field vector b^i which generate the expansion and rotation free geodetic magnetic field lines and electric field lines respectively, then the magnetic pressure is balanced by the electric pressure.

PROOF : Taking affine collineation w.r.t. the vectors a^i and b^i respectively and using the space-like counter part of Raychaudhuri's equation (4.4) w.r.t. a^i and b^i together with our assumption $\overset{*}{\theta} = \overset{*}{D}a^i = \overset{*}{w}^{ij} = \overset{\vee}{\theta} = \overset{\vee}{D}b^i = \overset{\vee}{w}^{ij} = 0$, we get the Theorem 4.2.

5. Conformal Motion

The infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i \delta_t$$

(δ_t is an infinitesimal parameter) is defined as

(1) conformal motion if

$$\mathcal{L}_\xi g_{ij} = \Lambda g_{ij}$$

where Λ is a non-zero scalar function and ξ^i is any arbitrary vector.

(2) motion if

$$\mathcal{L}_\xi g_{ij} = 0.$$

Theorem 5.1 — The conformal motion in the space-time of the relativistic electromagnetic fluid w.r.t. the magnetic field vector degenerates into motion when the magnetic field lines are free from the expansion, the rotation and the shear.

Proof : The conformal motion w.r.t. the magnetic field vector h^i is given by

$$h_{i;j} + h_{j;i} = \Lambda g_{ij}.$$

Consequently, we have

$$h^i_{;i} = 2\Lambda = -\ 2h^i Du_i. \qquad \ldots(5.1)$$

Using the fact

$$2h^i Du_i = 0 \Leftrightarrow \overset{*}{\theta} = \overset{*}{w}{}^{ij} = \overset{*}{\sigma}{}^{ij} = 0$$

in (5.1), we obtain

$$\Lambda = 0$$

which proves the statement.

Corollary 5.1 — The total pressure of the shear free relativistic electromagnetic fluid is balaced by the pressure due to magnetic field (magnetic pressure) along the magnetic field lines of constant magnetic permeability when the magnetic field lines are free from the expansion, the rotation and the shear.

Proof : Contracting (2.13) by $h^i = |\ h\ |\ a^i$ and using (1.1), (1.2), (1.4) and Theorem 5.1, we obtain Corollary 5.1.

Theorem 5.2 — The conformal motion in the space-time of the relativistic electromagnetic fluid w.r.t. the electric field vector degenerates into motion when the electric field lines are free from the expansion, the rotation and the shear.

The proof follows the pattern of the proof of Theorem 5.1.

Corollary 5.2 — The total pressure of the shear free relativistic electromagnetic fluid is balanced by the pressure due to electric field (electric pressure) along the electric field lines of constant dielectric capacity when the electric field lines are free from the expansion, the rotation and the shear.

The proof of Corollary 5.2 runs on the lines of the proof of Corollary 5.1.

Theorem 5.3 — The conformal motion in the space-time of the relativistic electromagnetic fluid w.r.t. the energy flux vector P^i degenerates into motion when the fluid is in steady state with rigid rotation.

Theorem 5.3 immediately follows from the Theorem 2.6 and the defining equation of conformal motion w.r.t. P^i.

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APPENDIX

$Du^i = u^i_{;j} u^j$, the normal curvature vector of the streamlines or the acceleration vector of the fluid element.

$\theta = u^i_{;i}$, the expansion of the streamlines.

$\sigma_{ij} = u_{(i;j)} - Du_{(i}u_{j)} - \frac{1}{3}\theta\gamma_{ij}$, the shear of the fluid.

$w_{ij} = u_{[i;j]} - Du_{[i}u_{j]}$, the rotation of the fluid.

$\gamma_{ij} = g_{ij} - u_i u_j$, the projection operator onto the 3-space quotient to the streamlines.

$\overset{*}{D}a^i = a^i_{;j}\, a^j$, the normal curvature vector of the magnetic field lines (magnetic flux vector).

$\overset{*}{\theta} = a^i_{;i}$, the expansion of the magnetic field lines.

$\overset{*}{\sigma}_{ij} = a_{(i;j)} + \overset{*}{D}a_{(i}a_{j)} - \frac{1}{3}\overset{*}{\theta}\overset{*}{\gamma}_{ij}$, the shear of the magnetic field lines.

$\overset{*}{w}_{ij} = a_{[i;j]} + \overset{*}{D}a_{[i}a_{j]}$, the rotation of the magnetic field lines.

$\overset{*}{\gamma}_{ij} = g_{ij} + a_i a_j$, the projection operator onto the 3-space quotient to the magnetic field lines.

$\overset{\vee}{D}b^i = b^i_{;j}\, b^j$, the normal curvature vector of the electric field lines (electric flux vector).

$\overset{\vee}{\theta} = b^i_{;i}$, the expansion of the electric field lines.

$\overset{\vee}{\sigma}_{ij} = b_{(i;j)} + \overset{\vee}{D}b_{(i}b_{j)} - \frac{1}{3}\overset{\vee}{\theta}\,\overset{\vee}{\gamma}_{ij}$, the shear of the electric field lines.

$\overset{\vee}{w}_{ij} = b_{[i;j]} + \overset{\vee}{D}b_{[i}b_{j]}$, the rotation of the electric field lines.

$\overset{\vee}{\gamma}_{ij} = g_{ij} + b_i b_j$, the projection operator onto the 3-space quotient to the electric field lines.