ON THE UNIVALENCE OF SOME ANALYTIC FUNCTIONS

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Let $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$, $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots$, and $h(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots$ be analytic and satisfy

(A) $$\text{Re}\left\{\frac{zf(z)}{\lambda zf(z) + (1-\lambda)\, g(z)\, h(z)}\right\} > 0$$

or

(B) $$\left|\frac{zf(z)}{\lambda zf(z) + (1-\lambda)\, g(z)\, h(z)} - 1\right| < 1$$

for $|z| < 1$, $0 \leqslant \lambda < 1$. We determine the values of R such that $f(z)$ is univalent and starlike for $|z| < R$ under the assumption that

(i) $\text{Re}\left\{\frac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $h(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$,

or

(ii) $\text{Re}\left\{\frac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $\text{Re}\left\{\frac{h(z)}{s_2(z)}\right\} > 0$ for $|z| < 1$, where $s_2(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$.

§1. A function $h(z) = z + c_2z^2 + \ldots$ is said to be starlike of order β, $0 \leqslant \beta \leqslant 1$, for $|z| < 1$ if $\text{Re}\left\{\frac{zh'(z)}{h(z)}\right\} > \beta$ for $|z| < 1$. If $\beta = 1$, then $h(z) = z$.

Ratti (1968) has obtained discs of univalence and starlikeness for certain classes of functions $f(z) = z + a_2z^2 + \ldots$ analytic in $|z| < 1$. Some of his results required that either

$$\text{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0 \quad \text{or} \quad \left|\frac{f(z)}{g(z)} - 1\right| < 1$$

for $|z| < 1$, where $\text{Re}\,\{g(z)/z\} = \text{Re}\,\{1 + b_2z + \ldots\} > 0$.

This was suggested in the theorems obtained by MacGregor (1963a, b). In a paper by Causey and Merkes (1970) the expression $g(z)/z$ was replaced by $g(z)/s_1(z)$, where $s_1(z)$ is starlike function of order α, $0 \leqslant \alpha \leqslant 1$.

Shah (1972) has obtained discs of univalence and starlikeness for certain classes of functions $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$ analytic in $|z| < 1$. Some of his results required that either

$$\text{(a)} \qquad \operatorname{Re}\left\{\frac{f(z)}{\lambda f(z) + (1-\lambda)\, g(z)}\right\} > 0$$

or

$$\text{(b)} \qquad \left|\frac{f(z)}{\lambda f(z) + (1-\lambda)\, g(z)} - 1\right| < 1$$

for $|z| < 1$, $0 \leqslant \lambda < 1$, where $\operatorname{Re}\{g(z)/z\} > 0$. Swamy (1977) has obtained discs of univalence and starlikeness for certain classes of functions $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$ analytic in $|z| < 1$. Some of his results required that either

$$\text{(A)} \qquad \operatorname{Re}\left\{\frac{zf(z)}{\lambda zf(z) + (1-\lambda)\, g(z)\, h(z)}\right\} > 0$$

or

$$\text{(B)} \qquad \left|\frac{zf(z)}{\lambda zf(z) + (1-\lambda)\, g(z)\, h(z)} - 1\right| < 1$$

for $|z| < 1$, $0 \leqslant \lambda < 1$, where

(i) $\operatorname{Re}\{g(z)/z\} > 0$, for $|z| < 1$, and $h(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$,

or

(ii) $\operatorname{Re}\{g(z)/z\} > 0$ for $|z| < 1$, and $\operatorname{Re}\{h(z)/z\} > 0$ for $|z| < 1$.

In this paper the expressions $g(z)/z$ and $h(z)/z$ are replaced by $g(z)/s_1(z)$ and $h(z)/s_2(z)$ respectively, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $s_2(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$.

§2. We require the following lemmas for our discussion :

Lemma 1 (Shah 1972) — If $P(z) = 1 + b_n z^n + b_{n+1}z^{n+1} + \ldots$ is analytic and satisfies $\operatorname{Re}(P(z)) > \alpha$, $0 \leqslant \alpha < 1$, for $|z| < 1$, then we have for $|z| < 1$

$$\left|\frac{zp'(z)}{p(z)}\right| \leqslant \frac{2n\,|z|^n(1-\alpha)}{(1-|z|^n)\,[1+(1-2\alpha)\,|z|^n]}.$$

Lemma 2 (Shah 1972) — Under the hypothesis of Lemma 1 we have for $|z| < 1$

$$\operatorname{Re}\,(p(z)) \geqslant \frac{1+(2\alpha-1)\,|z|^n}{1+|z|^n}.$$

Lemma 3 (Shah 1972) — If $\phi(z) = 1 + d_n z^n + d_{n+1} z^{n+1} + \ldots$ is analytic and $\text{Re}\,(\phi(z)) > 0$ for $|z| < 1$, then

$$[1 - \lambda\, |\phi(z)|\,]^{-1} \leqslant (1 - |z|^n)/[(1 - |z|^n) - \lambda(1 + |z|^n)]$$

for $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$, where $0 \leqslant \lambda < 1$.

Now we prove the following results.

§3. *Theorem* 1 — Suppose that $f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots$, $g(z) = z + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \ldots$, and $h(z) = z + c_{n+1} z^{n+1} + c_{n+2} z^{n+2} + \ldots$ are analytic for $|z| < 1$. Let $\text{Re}\left\{\dfrac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $h(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$. If $\text{Re}\left\{\dfrac{zf(z)}{\lambda z f(z) + (1 - \lambda)\, g(z)\, h(z)}\right\} > 0$ for $|z| < 1$, then $f(z)$ is univalent and starlike for

(i) $|z| < [(1 - \lambda)/(4n - 2n\lambda + \lambda + 1)]^{1/n}$ if $\alpha + \beta = 3/2$

and

(ii) $|z| < R^{1/n}$ if $\alpha + \beta \neq 3/2$

where

$$R = \frac{\{A^2 + 4(1 - \lambda^2)(2\alpha + 2\beta - 3)\}^{1/2} - A}{2(1 + \lambda)(2\alpha + 2\beta - 3)}$$

with

$$A = 4n - 2n\lambda + \lambda + 1 - (1 - \lambda)(2\alpha + 2\beta - 3).$$

PROOF : Let $\phi(z) = zf(z)/[\lambda z f(z) + (1 - \lambda)\, g(z)\, h(z)]$. Then $\phi(z)$ is analytic and $\text{Re}\,(\phi(z)) > 0$ for $|z| < 1$. Now

$$[1 - \lambda\phi(z)]\, zf(z) = (1 - \lambda)\, g(z)\, h(z)\, \phi(z). \qquad \ldots(1)$$

Multiplying the logarithmic derivative of both sides of equation (1) by z we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)} + \frac{zg'(z)}{g(z)} + \frac{z\phi'(z)/\phi(z)}{1 - \lambda\phi(z)} - 1. \qquad \ldots(2)$$

Equation (2) is valid for those z for which $1 - \lambda\phi(z) \neq 0$ and $|z| < 1$. Since $|\phi(z)| \leqslant (1 + |z|^n)/(1 - |z|^n)$, $1 - \lambda\phi(z) \neq 0$ in particular, if $|z| < [(1 - \lambda)/(1 + \lambda)]^{1/n}$. Let $p(z) = g(z)/s_1(z)$. So

$$\frac{zg'(z)}{g(z)} = \frac{zp'(z)}{p(z)} + \frac{zs_1'(z)}{s_1(z)}.$$

Therefore (2) reduces to

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)} + \frac{zs_1'(z)}{s_1(z)} + \frac{zP'(z)}{p(z)} + \frac{z\phi'(z)/\phi(z)}{1 - \lambda\phi(z)} - 1 \qquad \ldots(3)$$

and this gives

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} + \operatorname{Re}\left\{\frac{zs_1'(z)}{s_1(z)}\right\} - \left|\frac{zp'(z)}{p(z)}\right| - \left|\frac{z\phi'(z)/\phi(z)}{1-\lambda\phi(z)}\right| - 1. \qquad \ldots(4)$$

By using Lemma 2, Lemma 1 with $\alpha = 0$, and Lemma 3 we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1+(2\beta-1)|z|^n}{1+|z|^n} + \frac{1+(2\alpha-1)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{1-|z|^{2n}} - \frac{2n|z|^n}{(1-|z|^{2n})-\lambda(1+|z|^n)^2} - 1 \qquad \ldots(5)$$

provided that $|z| < [(1-\lambda)/(1+\lambda)]^{1/n}$.

After some simplification (5) reduces to the form

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{\begin{array}{c}(1+\lambda)(2\alpha+2\beta-3)|z|^{3n} + [2n\lambda + 4n + 1 + \lambda \\ - 2(2\alpha+2\beta-3)]|z|^{2n} \\ + [2n\lambda - 4n - 2 + (1-\lambda)(2\alpha+2\beta-3)]|z|^n + (1-\lambda)\end{array}}{(1-|z|^{2n})[(1-|z|^n)-\lambda(1+|z|^n)]}. \qquad \ldots(6)$$

Hence $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ if $|z| < [(1-\lambda)/(1+\lambda)]^{1/n}$ and if

$$\begin{aligned} G(|z|^n) = {} & (1+\lambda)(2\alpha+2\beta-3)|z|^{3n} \\ & + [4n + 2n\lambda + \lambda + 1 - 2(2\alpha+2\beta-3)]|z|^{2n} \\ & + [2n\lambda - 4n - 2 + (1-\lambda)(2\alpha+2\beta-3)]|z|^n \\ & + (1-\lambda) > 0. \end{aligned} \qquad \ldots(7)$$

Let $|z|^n = t$ and consider the cubic polynomial $G(t)$ for $0 \leqslant t \leqslant 1$. Since $G(0) = 1-\lambda > 0$, $G[(1-\lambda)/(1+\lambda)] = -4n\lambda(1-\lambda)/(1+\lambda)^2 < 0$ and $G(1) = 4n\lambda > 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1-\lambda)/(1+\lambda)$ and $G(t) > 0$ for $0 \leqslant t < t_1$ and $G(t) < 0$ for $t_1 < t < (1-\lambda)/(1+\lambda)$. Hence $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ for these z for which only the inequality (7) is true. Now the inequality (7) holds if, in particular

$$(1+\lambda)(2\alpha+2\beta-3)|z|^{3n} + [4n-2n\lambda+\lambda+1-2(2\alpha+2\beta-3)]|z|^{2n} + [2n\lambda-4n-2+(1-\lambda)(2\alpha+2\beta-3)]|z|^{n} + (1-\lambda) > 0$$

or

$$(|z|^{n}-1)[(1+\lambda)(2\alpha+2\beta-3)|z|^{2n} + \{4n-2n\lambda+\lambda+1-(1-\lambda)(2\alpha+2\beta-3)\}|z|^{n} + (\lambda-1)] > 0$$

or

$$(1+\lambda)(2\alpha+2\beta-3)|z|^{2n} + [4n-2n\lambda+\lambda+1-(1-\lambda)(2\alpha+2\beta-3)]|z|^{n} + (\lambda-1) < 0.$$

The last inequality holds if

$$|z|^{n} < (1-\lambda)/[4n-2n\lambda+\lambda+1]$$

when $\alpha+\beta = 3/2$ and

$$|z|^{n} < \frac{\{A^2+4(1-\lambda^2)(2\alpha+2\beta-3)\}^{1/2}-A}{2(1+\lambda)(2\alpha+2\beta-3)}$$

when $\alpha+\beta \neq 3/2$, where $A = 4n-2n\lambda+\lambda+1-(1-\lambda)(2\alpha+2\beta-3)$ and this completes the proof.

For $\alpha = 1$, the above result reduces to a result of Swamy (1977, Theorem 1). For $\alpha = 1, \beta = 1$, the above result reduces to a result of Shah (1972, Theorem 1). If we put $\lambda = 0$ in Theorem 1 we obtain the following result which, when $n = 1$, $\beta = 1$ reduces to a result of Causey and Merkes (1970, Theorem 3.1).

Corollary 1 — Suppose that $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$, $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots$, and $h(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots$ are analytic for $|z| < 1$. Let $\text{Re}\left\{\frac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $h(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$. If $\text{Re}\left\{\frac{zf(z)}{g(z)\,h(z)}\right\} > 0$ for $|z| < 1$, then $f(z)$ is univalent and starlike for

(i) $|z| < [1/(4n+1)]^{1/n}$ if $\alpha+\beta = 3/2$

and

(ii) $|z| < R^{1/n}$ if $\alpha+\beta \neq 3/2$

where

$$R = \frac{\{4n^2 + 8n + 1 + (\alpha + \beta)(\alpha + \beta - 4n - 2)\}^{1/2} - (2n + 2 - (\alpha + \beta))}{(2\alpha + 2\beta - 3)}. \quad ...(8)$$

The functions

$$f(z) = z(1 - z^n)^2/(1 + z^n)^{(4-2\alpha-2\beta+2n)/n},$$

$$g(z) = z(1 - z^n)/(1 + z^n)^{(2-2\alpha+n)/n},$$

$$s_1(z) = z/(1 + z^n)^{(2-2\alpha)/n},$$

and

$$h(z) = z/(1 + z^n)^{(2-2\beta)/n}$$

satisfy the hypothesis of Corollary 1 and the derivative of $f(z)$ vanishes at $z = [1/(4n + 1)]^{1/n}$ when $\alpha + \beta = 3/2$, and at $z = R^{1/n}$ when $\alpha + \beta \neq 3/2$, where R is as stated in (8). This shows that Corollary 1 is sharp and hence Theorem 1 is sharp at least for $\lambda = 0$.

Theorem 2 — Suppose that $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$, $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots$, and $h(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots$ are analytic for $|z| < 1$. Let $\text{Re}\left\{\frac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $\text{Re}\left\{\frac{h(z)}{s_2(z)}\right\} > 0$ for $|z| < 1$, where $s_2(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$. If $\text{Re}\left\{\frac{zf(z)}{\lambda zf(z) + (1-\lambda)g(z)h(z)}\right\} > 0$ for $|z| < 1$, then $f(z)$ is univalent and starlike for

(i) $|z| < [(1-\lambda)/(6n - 4n\lambda + \lambda + 1)]^{1/n}$ if $\alpha + \beta = 3/2$

and

(ii) $|z| < R^{1/n}$ if $\alpha + \beta \neq 3/2$

where

$$R = \frac{\{A^2 + 4(1 - \lambda^2)(2\alpha + 2\beta - 3)\}^{1/2} - A}{2(1 + \lambda)(2\alpha + 2\beta - 3)}$$

with

$$A = 6n - 4n\lambda + \lambda + 1 - (1 - \lambda)(2\alpha + 2\beta - 3).$$

PROOF : Let $p(z) = g(z)/s_1(z)$ and $q(z) = h(z)/s_2(z)$. Then proceeding exactly as in the proof of Theorem 1 we get

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \text{Re}\left\{\frac{zs_1'(z)}{s_1(z)}\right\} + \text{Re}\left\{\frac{zs_2'(z)}{s_2(z)}\right\} - \left|\frac{zp'(z)}{p(z)}\right| - \left|\frac{zq'(z)}{q(z)}\right| - \left|\frac{z\phi'(z)/\phi(z)}{1 - \lambda\phi(z)}\right| - 1.$$

Applying Lemma 2, Lemma 1 with $\alpha = 0$, and Lemma 3 we get

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1+(2\alpha-1)|z|^n}{1+|z|^n} + \frac{1+(2\beta-1)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{1-|z|^{2n}} - \frac{2n|z|^n}{1-|z|^{2n}} - \frac{2n|z|^n}{(1-|z|^{2n})-\lambda(1+|z|^n)^2} - 1. \quad \ldots(9)$$

provided that $|z| < [(1-\lambda)/(1+\lambda)]^{1/n}$. After some simplification (9) reduces to

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{\begin{aligned}&(1+\lambda)(2\alpha+2\beta-3)|z|^{3n} \\ &+[6n+4n\lambda+\lambda+1-2(2\alpha+2\beta-3)]|z|^{2n} \\ &+[4n\lambda-6n-2+(1-\lambda)(2\alpha+2\beta-3)]|z|^n+(1-\lambda)\end{aligned}}{(1-|z|^{2n})[(1-|z|^n)-\lambda(1+|z|^n)]}. \quad \ldots(10)$$

Hence

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

for those z for which $|z| < [(1-\lambda)/(1+\lambda)]^{1/n}$ and the right side of inequality (10) is greater than zero. The latter holds if

$$\begin{aligned}G(|z|^n) = {} & (1+\lambda)(2\alpha+2\beta-3)|z|^{3n} \\ &+[6n+4n\lambda+\lambda+1-2(2\alpha+2\beta-3)]|z|^{2n} \\ &+[4n\lambda-6n-2+(1-\lambda)(2\alpha+2\beta-3)]|z|^n \\ &+(1-\lambda) > 0. \end{aligned} \quad \ldots(11)$$

Let $|z|^n = t$ and consider the cubic polynomial $G(t)$ for $0 \leqslant t \leqslant 1$. Since $G(0) = 1-\lambda > 0$, $G[(1-\lambda)/(1+\lambda)] = -4n\lambda(1-\lambda)/(1+\lambda)^2 < 0$, and $G(1) = 8n\lambda > 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1-\lambda)/(1+\lambda)$ and $G(t) > 0$ for $0 \leqslant t < t_1$ and $G(t) < 0$ for $t_1 < t < (1-\lambda)/(1+\lambda)$. Hence $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ for those z for which only the inequality (11) holds. Now the inequality (11) holds if, in particular

$$\begin{aligned}&(1+\lambda)(2\alpha+2\beta-3)|z|^{3n} \\ &+[6n-4n\lambda+\lambda+1-2(2\alpha+2\beta-3)]|z|^{2n} \\ &+[4n\lambda-6n-2+(1-\lambda)(2\alpha+2\beta-3)]|z|^n \\ &+(1-\lambda) > 0\end{aligned}$$

or

$$(|z|^n - 1)\{(1+\lambda)(2\alpha + 2\beta - 3)|z|^{2n} + [6n - 4n\lambda + \lambda + 1 - (1-\lambda)(2\alpha + 2\beta - 3)]|z|^n + (\lambda - 1)\} > 0$$

or

$$(1+\lambda)(2\alpha + 2\beta - 3)|z|^{2n} + [6n - 4n\lambda + \lambda + 1 - (1-\lambda)(2\alpha + 2\beta - 3)]|z|^n + (\lambda - 1) < 0.$$

The last inequality holds if

$$|z|^n < [(1-\lambda)/(6n - 4n\lambda + \lambda + 1)] \text{ when } \alpha + \beta = 3/2$$

and

$$|z|^n < \frac{\{A^2 + 4(1-\lambda^2)(2\alpha + 2\beta - 3)\}^{1/2} - A}{2(1+\lambda)(2\alpha + 2\beta - 3)}$$

when $\alpha + \beta \neq 3/2$, where $A = 6n - 4n\lambda + \lambda + 1 - (1-\lambda)(2\alpha + 2\beta - 3)$ and this completes the proof.

For $\alpha = 1, \beta = 1$ the above result reduces to a result of Swamy (1977, Theorem 3). If we put $\lambda = 0$ in Theorem 2 we obtain the following result.

Corollary 2 — Suppose that $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$, $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots$, and $h(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots$ are analytic for $|z| < 1$. Let $\operatorname{Re}\left\{\frac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $\operatorname{Re}\left\{\frac{h(z)}{s_2(z)}\right\} > 0$ for $|z| < 1$, where $s_2(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$. If $\operatorname{Re}\left\{\frac{zf(z)}{g(z)h(z)}\right\} > 0$ for $|z| < 1$, then $f(z)$ is univalent and starlike for

(i) $|z| < [1/(6n+1)]^{1/n}$ if $\alpha + \beta = 3/2$

and

(ii) $|z| < R^{1/n}$ if $\alpha + \beta \neq 3/2$,

where

$$R = \frac{\{9n^2 + 12n + 1 + (\alpha + \beta)(\alpha + \beta - n - 2)\}^{1/2} + (\alpha + \beta - 3n - 2)}{(2\alpha + 2\beta - 3)}. \qquad \ldots(12)$$

The functions

$$f(z) = z(1 - z^n)^3/(1 + z^n)^{(4-2\alpha-2\beta+3n)/n},$$
$$g(z) = z(1 - z^n)/(1 + z^n)^{(2-2\alpha+n)/n},$$
$$s_1(z) = z/(1 + z^n)^{(2-2\alpha)/n},$$
$$h(z) = z(1 - z^n)/(1 + z^n)^{(2-2\beta+n)/n},$$

and

$$s_2(z) = z/(1 + z^n)^{(2-2\beta)/n}$$

satisfy the hypothesis of Corollary 2 and $f'(z)$ vanishes at $z = [1/(6n + 1)]^{1/n}$ when $\alpha + \beta = 3/2$ and at $z = R^{1/n}$, where R is as stated in (12), when $\alpha + \beta \neq 3/2$. This shows that Corollary 2 is sharp and hence Theorem 2 is sharp at least for $\lambda = 0$.

Theorem 3 — Suppose that $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots$, $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \dots$ and $h(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \dots$ are analytic for $|z| < 1$. Let $\mathrm{Re}\left\{\frac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $h(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$. If $\left|\frac{zf(z)}{\lambda zf(z) + (1-\lambda)\, g(z)\, h(z)} - 1\right| < 1$ for $|z| < 1$, then $f(z)$ is univalent and starlike for $|z| < R^{1/n}$, where R is the smallest positive root of the equation

$$\lambda(2\alpha + 2\beta - 3)\, R^3 + [2n\lambda - n + 1 + \lambda - 2(\alpha + \beta - 1)]\, R^2 + [2n\lambda - 3n + \lambda - 2 + 2(1-\lambda)(\alpha + \beta - 1)]\, R + (1 - \lambda) = 0. \qquad \dots(13)$$

PROOF: Let

$$\psi(z) = \frac{zf(z)}{\lambda zf(z) + (1-\lambda)\, g(z)\, h(z)} - 1. \qquad \dots(14)$$

By hypothesis $\psi(z)$ is analytic and $|\psi(z)| < 1$ for $|z| < 1$ and hence by a result of Goluzin (1945) we have that for $|z| < 1$

$$|\psi'(z)| \leqslant n\, |z|^{n-1} (1 - |\psi(z)|^2)/(1 - |z|^{2n}) \qquad \dots(15)$$

and by Schwarz's Lemma for $|z| < 1$

$$|\psi(z)| \leqslant |z|^n. \qquad \dots(16)$$

Now from (14) we obtain

$$[1 - \lambda(1 + \psi(z))]\, zf(z) = (1 - \lambda)\, g(z)\, h(z)\, (1 + \psi(z)). \qquad \dots(17)$$

Multiplying the logarithmic derivative of both sides of (17) by z we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)} + \frac{zg'(z)}{g(z)} + \frac{z\psi'(z)}{(1 + \psi(z))\,(1 - \lambda - \lambda\psi(z))} - 1. \qquad \dots(18)$$

If we let $g(z) = p(z)\, s_1(z)$, then we have from (18)

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)} + \frac{zs_1'(z)}{s_1(z)} + \frac{zp'(z)}{p(z)} + \frac{z\psi'(z)}{(1+\psi(z))\,(1-\lambda-\lambda\psi(z))} - 1$$

and this gives

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \text{Re}\left\{\frac{zh'(z)}{h(z)}\right\} + \text{Re}\left\{\frac{zs_1'(z)}{s_1(z)}\right\} - \left|\frac{zp'(z)}{p(z)}\right| - \left|\frac{z\psi'(z)}{(1+\psi(z))\,(1-\lambda-\lambda\psi(z))}\right| - 1.$$

Applying Lemma 2, Lemma 1 with $\alpha = 0$, we get in view of (15) for $|z| < 1$

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1+(2\beta-1)|z|^n}{1+|z|^n} + \frac{1+(2\alpha-1)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{1-|z|^{2n}} - \frac{n|z|^n(1-|\psi(z)|^2)}{(1-|z|^{2n})\,|1+\psi(z)|\;|1-\lambda-\lambda\psi(z)|} - 1$$

$$\geqslant \frac{2+(2\alpha+2\beta-2)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{1-|z|^{2n}} - \frac{n|z|^n(1+|\psi(z)|)}{(1-|z|^{2n})\,|1-\lambda-\lambda\psi(z)|} - 1.$$

By using (16), we have

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{2+(2\alpha+2\beta-2)|z|^n}{1+|z|^n} - \frac{2n|z|^n}{1-|z|^{2n}} - \frac{n|z|^n}{(1-|z|^n)(1-\lambda-\lambda|z|^n)} - 1$$

$$= \frac{\lambda(2\alpha+2\beta-3)|z|^{3n} + [2n\lambda-n+1+\lambda-2(\alpha+\beta-1)]|z|^{2n} + [2n\lambda-3n+\lambda-2+2(1-\lambda)(\alpha+\beta-1)]|z|^n + (1-\lambda)}{(1-|z|^{2n})(1-\lambda-\lambda|z|^n)}$$

valid for $|z| < [(1-\lambda)/\lambda]^{1/n}$. Hence $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ if $|z| < [(1-\lambda)/\lambda]^{1/n}$ and if

$$\begin{aligned} G(|z|^n) = {} & \lambda(2\alpha+2\beta-3)|z|^{3n} \\ & + [2n\lambda-n+1+\lambda-2(\alpha+\beta-1)]|z|^{2n} \\ & + [2n\lambda-3n+\lambda-2+2(1-\lambda)(\alpha+\beta-1)]|z|^n \\ & + (1-\lambda) > 0. \end{aligned} \qquad \text{...(19)}$$

Let $|z|^n = t$ and consider the cubic polynomial $G(t)$ for $0 \leqslant t \leqslant 1$. Since $G(0) = 1-\lambda > 0$, and $G[(1-\lambda)/\lambda] = -[n(1-\lambda)]/\lambda^2 < 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1-\lambda)/\lambda$ and $G(t) > 0$ for $0 \leqslant t < t_1$

and $G(t) < 0$ for $t_1 < t < (1 - \lambda)/\lambda$. Hence $f(z)$ is univalent and starlike for $|z| < R^{1/n}$, in view of inequality (19), where R is the smallest positive root of the equation (13).

For $\alpha = 1$ the above result reduces to a result of Swamy (1977, Theorem 4). For $\alpha = 1$, $\beta = 1$ the above result reduces to a result of Shah (1972, Theorem 4). If we put $\lambda = 0$ in Theorem 3 we obtain the following result which, when $n = 1$, $\beta = 1$ reduces to a result of Causey and Merkes (1970, Theorem 3.3).

Corollary 3 — Suppose $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$, $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots$, and $h(z) = z + c_{n+1}z^{n+1}c_{n+2}z^{n+2} + \ldots$ are analytic for $|z| < 1$. Let Re $\left\{\frac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $h(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$. If $\left|\frac{zf(z)}{g(z)\,h(z)} - 1\right| < 1$ for $|z| < 1$, then $f(z)$ is univalent and starlike for

(i) $|z| < 1/5$ if $\alpha + \beta = 1$ and $n = 1$

and

(ii) $|z| < R^{1/n}$ if $\alpha + \beta \neq 1$

where

$$R = \frac{\{9n^2 + 28n + 4 + 4(\alpha + \beta)\,[\alpha + \beta - (2 + 3n)]\}^{1/2} + (2(\alpha + \beta - 4 - 3n))}{2(2(\alpha + \beta) + n - 3)}. \qquad \ldots(20)$$

The functions

$$f(z) = z(1 - z^n)^2/(1 + z^n)^{(4-2\alpha-2\beta+n)/n},$$

$$g(z) = z(1 - z^n)/(1 + z^n)^{(2-2\alpha+n)/n},$$

$$s_1(z) = z/(1 + z^n)^{(2-2\alpha)/n},$$

and

$$h(z) = z/(1 + z^n)^{(2-2\beta)/n}$$

satisfy the hypothesis of Corollary 3 and the derivative of $f(z)$ vanishes at $z = 1/5$ when $\alpha + \beta = 1$ and $n = 1$, and at $z = R^{1/n}$, where R is as stated in (20). This shows that Corollary 3 is sharp and hence Theorem 3 is sharp at least for $\lambda = 0$.

Theorem 4 — Suppose $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$, $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots$, and $h(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots$ are analytic for $|z| < 1$. Let Re $\left\{\frac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and Re $\left\{\frac{h(z)}{s_2(z)}\right\} > 0$ for $|z| < 1$, where $s_2(z)$ is starlike of order β,

$0 \leqslant \beta \leqslant 1$. If $\left|\frac{zf(z)}{\lambda zf(z) + (1-\lambda)\, g(z)\, h(z)} - 1\right| < 1$ for $|z| < 1$, then $f(z)$ is univalent and starlike for $|z| < R^{1/n}$, where R is the smallest positive root of the equation

$$\lambda(2\alpha + 2\beta - 3) R^3 + [4n\lambda - n + 1 + \lambda - 2(\alpha + \beta - 1)] R^2 + [4n\lambda - 5n + \lambda - 2 + 2(1-\lambda)(\alpha + \beta - 1)] R + (1 - \lambda) = 0. \quad \ldots(21)$$

PROOF : Let $p(z) = g(z)/s_1(z)$ and $q(z) = h(z)/s_2(z)$. Therefore,

$$\frac{zg'(z)}{g(z)} = \frac{zp'(z)}{p(z)} + \frac{zs_1'(z)}{s_1(z)} \quad \ldots(22)$$

and

$$\frac{zh'(z)}{h(z)} = \frac{zq'(z)}{q(z)} + \frac{zs_2'(z)}{s_2(z)}. \quad \ldots(23)$$

Now proceeding as in the proof of Theorem 3, we get

$$\frac{zf'(z)}{f(z)} = \frac{zh'(z)}{h(z)} + \frac{zg'(z)}{g(z)} + \frac{z\psi'(z)}{(1+\psi(z))(1-\lambda-\lambda\psi(z))} - 1.$$

By using (22) and (23) we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zs_1'(z)}{s_1(z)} + \frac{zs_2'(z)}{s_2(z)} + \frac{zp'(z)}{p(z)} + \frac{zq'(z)}{q(z)} + \frac{z\psi'(z)}{(1+\psi(z))(1-\lambda-\lambda\psi(z))} - 1$$

and this gives

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \operatorname{Re}\left\{\frac{zs_1'(z)}{s_1(z)}\right\} + \operatorname{Re}\left\{\frac{zs_2'(z)}{s_2(z)}\right\} - \left|\frac{zp'(z)}{p(z)}\right| - \left|\frac{zq'(z)}{q(z)}\right| - \left|\frac{z\psi'(z)}{(1+\psi(z))(1-\lambda-\lambda\psi(z))}\right| - 1.$$

Applying Lemma 2, Lemma 1 with $\alpha = 0$, we get in views of (15) and (16)

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1 + (2\alpha - 1)|z|^n}{1 + |z|^n} + \frac{1 + (2\beta - 1)|z|^n}{1 + |z|^n} - \frac{2n|z|^n}{1 - |z|^n} - \frac{2n|z|^n}{1 - |z|^n} - \frac{n|z|^n}{(1 - |z|^n)(1 - \lambda - \lambda|z|^n)} - 1 \quad \ldots(24)$$

valid for $|z| < [(1-\lambda)/\lambda]^{1/n}$.

After some simplification (24) reduces to

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{\lambda(2\alpha+2\beta-3)\,|z|^{3n}+[4n\lambda-n+1+\lambda-2(\alpha+\beta-1)]\,|z|^{2n} + [4n\lambda-5n+\lambda-2+2(1-\lambda)(\alpha+\beta-1)]\,|z|^{n}+(1-\lambda)}{(1-|z|^{2n})[1-\lambda-\lambda|z|^{n}]}. \quad \ldots(25)$$

Hence

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

for those z for which $|z| < [(1-\lambda)/\lambda]^{1/n}$ and the right side of inequality (25) is greater than zero. The later holds if

$$\begin{aligned} G(|z|^{n}) = {} & \lambda(2\alpha+2\beta-3)\,|z|^{3n} \\ & + [4n\lambda-n+1+\lambda-2(\alpha+\beta-1)]\,|z|^{2n} \\ & + [4n\lambda-5n+\lambda-2+2(1-\lambda)(\alpha+\beta-1)]\,|z|^{n} \\ & + (1-\lambda) > 0. \end{aligned} \quad \ldots(26)$$

Let $|z|^{n} = t$ and consider the cubic polynomial $G(t)$ for $0 \leqslant t \leqslant 1$. Since $G(0) = 1-\lambda > 0$ and $G[(1-\lambda)/\lambda] = -(n(1-\lambda))/\lambda^2 < 0$, it follows that $G(t_1) = 0$ for some t_1 such that $0 < t_1 < (1-\lambda)/\lambda$ and $G(t) > 0$ for $0 \leqslant t < t_1$ and $G(t) < 0$ for $t_1 < t < (1-\lambda)/\lambda$. Hence $f(z)$ is univalent and starlike for $|z| < R^{1/n}$, in view of inequality (26), where R is the smallest positive root of the equation (21).

For $\alpha = 1, \beta = 1$ the above result reduces to a result of Swamy (1977, Theorem 6). If we put $\lambda = 0$ in Theorem 4 we obtain the following result.

Corollary 4 — Suppose $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$, $g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots$, and $h(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots$ are analytic for $|z| < 1$. Let $\operatorname{Re}\left\{\frac{g(z)}{s_1(z)}\right\} > 0$ for $|z| < 1$, where $s_1(z)$ is starlike of order α, $0 \leqslant \alpha \leqslant 1$, and $\operatorname{Re}\left\{\frac{h(z)}{s_2(z)}\right\} > 0$ for $|z| < 1$, where $s_2(z)$ is starlike of order β, $0 \leqslant \beta \leqslant 1$. If $\left|\frac{zf(z)}{g(z)h(z)} - 1\right| < 1$ for $|z| < 1$, then $f(z)$ is univalent and starlike for

(i) $|z| < 1/7$ if $\alpha + \beta = 1$ and $n = 1$

and

(ii) $|z| < R^{1/n}$ if $\alpha + \beta \neq 1$

where

$$R = \frac{\{25n^2 + 44n + 4 + 4(\alpha + \beta)(\alpha + \beta - 2 - 5n)\}^{1/2} + [2(\alpha + \beta) - 4 - 5n]}{2[2(\alpha + \beta) + n - 3]}. \quad \ldots(27)$$

The functions

$$f(z) = z(1 - z^n)^3/(1 + z^n)^{(4-2\alpha-2\beta+2n)/n},$$

$$g(z) = z(1 - z^n)/(1 + z^n)^{(2-2\alpha+n)/n},$$

$$s_1(z) = z/(1 + z^n)^{(2-2\alpha)/n},$$

$$h(z) = z(1 - z^n)/(1 + z^n)^{(2-2\beta+n)/n},$$

and

$$s_2(z) = z/(1 + z^n)^{(2-2\beta)/n}$$

satisfy the hypothesis of Corollary 4 and $f'(z)$ vanishes at $z = 1/7$ when $\alpha + \beta = 1$ and $n = 1$, and at $z = R^{1/n}$ when $\alpha + \beta \neq 1$, where R is as stated in (27). This shows that Corollary 4 is sharp and hence Theorem 4 is sharp at least for $\lambda = 0$.

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