# UNSTEADY CHANNEL FLOW OF AN ELASTICO-VISCOUS LIQUID

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A boundary-initial value problem of an unsteady slow flow of an incompressible elastico-viscous fluid (Oldroyd $B$-liquid) induced by the elliptic harmonic oscillations of two infinite parallel disks has been investigated. Laplace transform technique has been used to solve the problem. On taking the motion considerably long time after the oscillations ($A_1 + B_1 \neq 0, C_1 + D_1 \neq 0$) of the disks, the transients have the time to die down. The damping of oscillations towards the normal to the disks is smaller in elastico-viscous liquid in comparison to a liquid having the same viscosity but no elasticity and also the rate of change of phase in viscoelastic liquid is quicker. The effect of elasticity on the flow pattern is to produe further disturbances partly of the same and partly of higher harmonic.

## NOMENCLATURE

$x, y$ = distance measured along and perpendicular to the disks

$u, v$ = components of velocity along and perpendicular to the disks

$\mu, \nu$ = coefficient of viscosity and kinematic viscosity

$\frac{D}{Dt}, \lambda_1, \lambda_2$ = Jaunmann derivative, stress relaxation time, rate of strain retardation time

$\sigma$ = elastic parameter

$S_{ij}, p_{ij}, d_{ij}$ = stress tensor, deviatoric stress tensor, rate of strain tensor

$W_{ij}, g_{ij}$ = vorticity tensor and metric tensor

$p$ = isotropic mean pressure.

The physical significance of $\lambda_1$ is that if the motion is stopped suddenly the stresses will decay as exp $(-t/\lambda_1)$ and the physical singificance of $\lambda_2$ is that if the stresses are removed the motion will decay as exp $(-t/\lambda_2)$.

## 1. INTRODUCTION

Debnath (1972) has investigated the problem of an unsteady hydromagnetic boundary layer flow induced in an incompressible, electrically conducting inviscid fluid between two infinite parallel disks in the presence of a uniform magnetic field by the elliptic harmonic oscillations of the disks. Johri (1974, 1975) has recently discussed the problem of an elastico-viscous flow (Rivlin-Ericksen model and Maxwell model) induced by circular oscillations of two infinite parallel disks.

In the present investigation we have studied the problem of unsteady slow flow of Oldroyd $B$-Liquid between two infinite parallel disks. The flow is induced by the elliptic harmonic oscillations of the disks.

Many authors have discussed the various models, proposed to describe the mechanical behaviour of viscoelastic materials, and concluded that of the relative simple ones, Oldroyd's liquid $B$ model is the most reasonable one to represent viscoelastic liquids at the present time. The constitutive equation of such a liquid, as given by Oldroyd is

$$\left.\begin{aligned} S_{ij} &= -pg_{ij} + p_{ij} \\ p_{ij} + \lambda_1 \frac{D}{Dt}(p_{ij}) &= 2\mu\left[d_{ij} + \lambda_2 \frac{D}{Dt}(d_{ij})\right] \end{aligned}\right\} \quad \ldots(1)$$

where

$$\frac{D}{Dt}(b_{ij}) = \frac{\partial}{\partial t} b_{ij} + u^k b_{ij,k} - W_{ik} b^k_j + W_{kj} b^k_i$$

$$W_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}), \quad d_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

$$\frac{D}{Dt}(g_{ij}) = 0, \ \frac{D}{Dt}(g^{ij}) = 0$$

and momentum equation in the absence of extraneous forces is

$$\rho\left[\frac{\partial u_i}{\partial t} + u_{i,j} u^j\right] = -p_{,j} g^{ij} + p_{ij,j}. \quad \ldots(2)$$

The flow is generated by the non-torsional oscillations of the disks given by

$$\left.\begin{aligned} u &= f(t) = Ae^{i\omega_1 t} + Be^{-i\omega_1 t}, \text{ on } y = 0, \text{ for } t > 0 \\ u &= g(t) = Ce^{i\omega_2 t} + De^{-i\omega_2 t}, \text{ on } y = h, \text{ for } t > 0 \end{aligned}\right\} \quad \ldots(3)$$

where $A, B, C, D$ are complex constants and $\omega_1$, $\omega_2$ are imposed oscillations. For a fluid at rest for all $t < 0$ it may be assumed that the initial state of stress is zero. The initial conditions are :

$$u(y, t) = 0, \frac{\partial u}{\partial t} = 0 \text{ at } t = 0 \text{ and for all } y. \quad \ldots(4)$$

Taking the motion to be slow, eqns. (1) and (2) are reduced to

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t} = \nu\left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2} \quad \ldots(5)$$

$$\left(\nu = \frac{\mu}{\rho}\right)$$

Writing
$$\left.\begin{aligned} \bar{u} &= \frac{u}{U_0\lambda_1^2},\ \bar{y} = \frac{y}{\sqrt{\nu\lambda_1}},\ \bar{t} = \frac{t}{\lambda_1} \\ \sigma &= \frac{\lambda_2}{\lambda_1},\ \bar{h} = \frac{h}{\sqrt{\nu\lambda_1}},\ \sigma_1 = \omega_1\lambda_1,\ \sigma_2 = \omega_2\lambda_1 \end{aligned}\right\} \quad \ldots(6)$$

and substituting these in eqn. (5), we get

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \sigma \frac{\partial^3 \bar{u}}{\partial \bar{y}^2 \partial \bar{t}}. \quad \ldots(7)$$

Also, the boundary and initial conditions will be respectively :

$$\left.\begin{aligned} \bar{u} &= A_1 e^{i\sigma_1 \bar{t}} + B_1 e^{-i\sigma_1 \bar{t}} \text{ on } \bar{y} = 0 \text{ for } \bar{t} > 0 \\ \bar{u} &= C_1 e^{i\sigma_2 \bar{t}} + D_1 e^{-i\sigma_2 \bar{t}} \text{ on } \bar{y} = \bar{h} \text{ for } \bar{t} > 0 \end{aligned}\right\} \quad \ldots(8)$$

$$\bar{u}(\bar{y}, \bar{t}) = 0,\ \frac{\partial \bar{u}}{\partial \bar{t}} = 0 \text{ at } \bar{t} = 0 \text{ and for all } \bar{y}. \quad \ldots(9)$$

## 2. Method of Solution

Applying the Laplace transform in (7) subject to (8) and (9), it turns out that the solution of the problem can readily be represented by the Laplace inversion integral in the form:

$$\bar{u}(\bar{y}, \bar{t}) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \left\{ \left( \frac{C_1}{p - i\sigma_2} + \frac{D_1}{p + i\sigma_2} \right) \frac{\sinh \sqrt{\frac{p(1+p)}{(1+\sigma p)}}\, \bar{y}}{\sinh \sqrt{\frac{p(1+p)}{(1+\sigma p)}}\, \bar{h}} - \left( \frac{A_1}{p - i\sigma_1} + \frac{B_1}{p + i\sigma_1} \right) \frac{\sinh \sqrt{\frac{p(1+p)}{(1+\sigma p)}}\, (\bar{y} - \bar{h})}{\sinh \sqrt{\frac{p(1+p)}{(1+\sigma p)}}\, \bar{h}} \right\} e^{p\bar{t}}\, dp. \quad \ldots(10)$$

Evaluation of these integrals requires a knowledge of the nature and location of the singularities of the integral (10). Evidently, the integrand of (10) is a single valued function of $p$ and has simple poles at $p = \pm i\sigma_1$, $p = \pm i\sigma_2$ and at $p = p_n$, where $p_n$ is given by

$$\left.\begin{aligned} p_n &= \tfrac{1}{2}\left[-(1 + \sigma R_n) \pm \sqrt{(1 + \sigma R_n)^2 - 4R_n}\right] \\ R_n &= \frac{n^2\pi^2}{\bar{h}^2},\ n = 0, 1, 2, \ldots \end{aligned}\right\} \quad \ldots(11)$$

It may be observed that for a given value of $n$, the poles given by (11) are negative real numbers if $(1 + \sigma R_n)^2 > 4R_n$; any poles for which $(1 + \sigma R_n)^2 < 4R_n$

are complex conjugates which lie on the circle $|p| = \sqrt{R_n}$. These poles are simple poles, but when $(1 + \sigma R_n)^2 = 4R_n$, a set of double poles occurs at

$$p = p_n = -\left(\frac{1 + \sigma R_n}{2}\right) = -\sqrt{R_n}.$$

The Bromwich contour integral can be evaluated by using the theory of residues when all the singularities are simple poles, it turns out from the computation of residues that the velocity field $\bar{u}(\bar{y}, \bar{t})$ is

$$\bar{u}(\bar{y},\bar{t}) = A_1 e^{i\sigma_1 \bar{t}} \frac{\sinh \sqrt{\frac{i\sigma_1(1+i\sigma_1)}{1+i\sigma\sigma_1}}(\bar{y}-\bar{h})}{\sinh\sqrt{\frac{i\sigma_1(1+i\sigma_1)}{(1+i\sigma\sigma_1)}}\bar{h}} + B_1 e^{-i\sigma_1\bar{t}}$$

$$\times \frac{\sinh \sqrt{\frac{i\sigma_1(i\sigma_1-1)}{1-i\sigma\sigma_1}}(\bar{y}-\bar{h})}{\sinh\sqrt{\frac{i\sigma_1(i\sigma_1-1)}{1-i\sigma\sigma_1}}\bar{h}} + C_1 e^{i\sigma_2\bar{t}} \frac{\sinh\sqrt{\frac{i\sigma_2(1+i\sigma_2)}{1+i\sigma\sigma_2}}\bar{y}}{\sinh\sqrt{\frac{i\sigma_2(1+i\sigma_2)}{1+i\sigma\sigma_2}}\bar{h}}$$

$$+ D_1 e^{-i\sigma_2\bar{t}} \frac{\sinh\sqrt{\frac{i\sigma_2(i\sigma_2-1)}{1-i\sigma\sigma_2}}\bar{y}}{\sinh\sqrt{\frac{i\sigma_2(i\sigma_2-1)}{1-i\sigma\sigma_2}}\bar{h}}$$

$$+ \sum_{n=0}^{\infty} \frac{2(-1)^n (C_1 + D_1) p_n (1+\sigma p_n)^{3/2} \sqrt{p_n(1+p_n)}\, e^{p_n \bar{t}}}{\bar{h}(p_n^2 + \sigma_2^2)\{(1+2p_n)\sqrt{1+\sigma p_n} - \sigma\sqrt{p_n(1+p_n)}\}} \sin\frac{n\pi\bar{y}}{\bar{h}}$$

$$+ \frac{2(A_1+B_1)}{\bar{h}} \sum_{n=0}^{\infty} \frac{(-1)^n p_n (1+\sigma p_n)^{3/2} \sqrt{p_n(1+p_n)}\, e^{p_n\bar{t}} \sin\frac{n\pi(\bar{y}-\bar{h})}{\bar{h}}}{(p_n^2+\sigma_1^2)\{(1+2p_n)\sqrt{1+\sigma p_n} - \sigma\sqrt{(1+p_n)p_n}\}}$$

$$= \bar{u}_{st.}(\bar{y},\bar{t}) + \bar{u}_{tr.}(\bar{y},\bar{t}). \qquad \ldots(12)$$

The first four terms of (12) represent steady state velocity field and are thus denoted by $\bar{u}_{st.}(\bar{y}, \bar{t})$. The last two infinite sums correspond to the transient component of the solution and are represented by $\bar{u}_{tr.}(\bar{y}, \bar{t})$.

For the case of double pole occurring to $p = -\frac{n\pi}{\bar{h}}$ $(n = 0, 1, 2, \ldots)$, the velocity distribution is given by

$$\bar{u}(\bar{y}, \bar{t}) = \bar{u}_{st.}(\bar{y}, \bar{t}) + \sum_{n=0}^{\infty} L \exp\left(-\frac{n\pi\bar{t}}{\bar{h}}\right) + \sum_{n=0}^{\infty} tM \exp\left(-\frac{n\pi\bar{t}}{\bar{h}}\right) \qquad \ldots(13)$$

where

$$\left.\begin{aligned} L &= \left[\frac{d}{dp}\{f(p)\}\left(p+\frac{n\pi}{\bar{h}}\right)^2\right]_{p=-n\pi/\bar{h}} \\ M &= \left[\left(p+\frac{n\pi}{\bar{h}}\right)^2 f(p)\right]_{p=-n\pi/\bar{h}} \\ f(p) &= \left(\frac{C_1}{p-i\sigma_2}+\frac{D_1}{p+i\sigma_2}\right)\frac{\sinh k\bar{y}}{\sinh k\bar{h}} \\ &\quad -\left(\frac{A_1}{p-i\sigma_1}+\frac{B_1}{p+i\sigma_1}\right)\frac{\sinh k(\bar{y}-\bar{h})}{\sinh k\bar{h}} \\ k &= \sqrt{\frac{p(1+p)}{(1+\sigma p)}}. \end{aligned}\right\} \qquad \ldots(14)$$

A notable feature of (12) - (13) is that if $A_1 + B_1 \neq 0$ and $C_1 + D_1 \neq 0$, the transient effects die out exponentially in the limit $t \to \infty$ and the final steady solution [consists of first four terms in (12) or (13)] is reached in the limit and represents the boundary layers on disks, where $\frac{\sigma_1}{\sigma_2} = \frac{\omega_1}{\omega_2} \leqslant 1$. But when $A_1 + B_1 = 0$ or $C_1 + D_1 = 0$, the transient effects will decay to zero exponentially in the limit $t \to \infty$ or drop out automatically. Thus the steady state is eventually reached in either case.

## 3. Particular Cases

(I) $C_1 = 0$, $D_1 = 0$ (when $\omega_1 = \omega_2$ and $\bar{h} \to \infty$).

In that case from (12), we have

$$\bar{u}(\bar{y}, \bar{t}) = A_1 e^{-s(\sigma_1)\bar{y}+i\sigma_1\bar{t}} + B_1 e^{-s(-\sigma_1)\bar{y}-i\sigma_1\bar{t}}. \qquad \ldots(15)$$

This is actually the solution for the flow induced by the elliptic harmonic oscillations of a single infinite disk. This represents steady state velocity field. The effect of elastic parameter ($\sigma$) is to increase the velocity along the flow direction and the velocity decreases normal to the flow directions.

(II) For non-oscillatory case ($\omega_1 = 0 = \omega_2$), the velocity field is given by

$$\bar{u}(\bar{y}, \bar{t}) = (C_1 + D_1)\frac{\bar{y}}{\bar{h}} + (A_1 + B_1)\left(1 - \frac{\bar{y}}{\bar{h}}\right)$$

$$+ \frac{2(C_1 + D_1)}{\bar{h}} \sum_{n=0}^{\infty} \frac{(-1)^n p_n(1 + \sigma p_n)^{3/2} \sqrt{p_n(1 + p_n)} \sin \frac{n\pi\bar{y}}{\bar{h}}}{(p_n^2 + \sigma_2^2)\{(1 + 2p_n)\sqrt{1 + \sigma p_n} - \sigma\sqrt{p_n(1 + p_n)}\}}$$

$$+ \frac{2(A_1 + B_1)}{\bar{h}} \sum_{n=0}^{\infty} \frac{(-1)^n p_n(1 + \sigma p_n)^{3/2} \sqrt{p_n(1 + p_n)} \sin \frac{n\pi(\bar{y} - \bar{h})}{\bar{h}}}{(p_n^2 + \sigma_1^2)\{(1 + 2p_n)\sqrt{1 + \sigma p_n)} - \sigma\sqrt{p_n(1 + p_n)}\}} \quad \ldots(16)$$

whence the steady solution has the form :

$$\bar{u}_{st.}(\bar{y}, \bar{t}) = (A_1 + B_1)\left(1 - \frac{\bar{y}}{\bar{h}}\right) + (C_1 + D_1)\frac{\bar{y}}{\bar{h}} \cdot \quad \ldots(17)$$

This solution describes a steady plane Coutte flow and is identical with the corresponding result in a viscous flow.

## 4. Skin Friction

The skin friction on the disks is given by (Mishra and Dash 1972)

$$\tau_{w(\bar{y}-0)} = \left\{\bar{\mu}\left[1 - (\lambda_1 - \lambda_2)\frac{\partial}{\partial t}\right]\frac{\partial u}{\partial y}\right\}_{y=0}$$

$$= \bar{\mu}\left[\left\{1 - (1 - \sigma)\frac{\partial}{\partial \bar{t}}\right\}\frac{\partial \bar{u}}{\partial \bar{y}}\right]_{\bar{y}=0}$$

$$= A_1 e^{i\sigma_1\bar{t}} s(\sigma_1) \coth s(\sigma_1)\bar{h}\{1 - (1 - \sigma) i\sigma_1\}$$

$$+ B_1 s(-\sigma_1) e^{-i\sigma_1\bar{t}}\{1 + (1 - \sigma) i\sigma_1\} \coth s(-\sigma_1)\bar{h}$$

$$+ \frac{C_1 s(\sigma_2)\{1 - (1 - \sigma) i\sigma_2\} e^{i\sigma_2\bar{t}}}{\sinh s(\sigma_2)\bar{h}}$$

$$+ \frac{D_1 s(-\sigma_2)\{1 + (1 - \sigma) i\sigma_2\} e^{-i\sigma_2\bar{t}}}{\sinh s(-\sigma_2)\bar{h}}$$

$$+ \frac{2\pi(C_1 + D_1)}{\bar{h}^2} \sum_{n=0}^{\infty} (-1)^n s(p_n, \sigma_2)\{1 - (1 - \sigma) p_n\} e^{p_n\bar{t}} +$$

*(equation continued on p. 487)*

$$+ \frac{2\pi}{\bar{h}^2} (A_1 + B_1) \sum_{n=0}^{\infty} (-1)^n s(p_n, \sigma_1) \{1 - (1-\sigma) p_n\} e^{p_n \bar{t}} \qquad \ldots(18)$$

and

$$\tau_{\omega(\bar{y}=\bar{h})} = \bar{\mu}\left[\left\{1 - (1-\sigma)\frac{\partial}{\partial \bar{t}}\right\}\frac{\partial \bar{u}}{\partial \bar{y}}\right]_{\bar{y}=\bar{h}}$$

$$= \frac{A_1 s(\sigma_1)\{1 - (1-\sigma) i\sigma_1\} e^{i\sigma_1 \bar{t}}}{\sinh s(\sigma_1) \bar{h}}$$

$$+ \frac{B_1 s(-\sigma_1)\{1 + (1-\sigma) i\sigma_1\} e^{-i\sigma_1 \bar{t}}}{\sinh s(-\sigma_1) \bar{h}}$$

$$+ C_1 e^{i\sigma_2 \bar{t}} s(\sigma_2) \coth s(\sigma_2) \bar{h} \{1 - (1-\sigma) i\sigma_2\}$$

$$+ D_1 e^{-i\sigma_2 \bar{t}} s(-\sigma_2) \{1 + (1-\sigma) i\sigma_2\} \coth s(-\sigma_2) \bar{h}$$

$$+ \frac{2\pi(C_1 + D_1)}{\bar{h}^2} \sum_{n=0}^{\infty} s(p_n, \sigma_2) e^{p_n \bar{t}} \{1 - (1-\sigma) p_n\}$$

$$+ \frac{2\pi(A_1 + B_1)}{\bar{h}^2} \sum_{n=0}^{\infty} s(p_n, \sigma_1) \{1 - (1-\sigma) p_n\} e^{p_n \bar{t}} \qquad \ldots(19)$$

where

$$\left.\begin{aligned} s(\sigma_1) &= \sqrt{\frac{i\sigma_1(1 + i\sigma_1)}{1 + i\sigma\sigma_1}}, \quad s(-\sigma_1) = \sqrt{\frac{i\sigma_1(i\sigma_1 - 1)}{1 - i\sigma\sigma_1}} \\ s(\sigma_2) &= \sqrt{\frac{i\sigma_2(1 + i\sigma_2)}{1 + i\sigma\sigma_2}}, \quad s(-\sigma_2) = \sqrt{\frac{i\sigma_2(i\sigma_2 - 1)}{1 - i\sigma\sigma_2}} \end{aligned}\right\} \qquad \ldots(20)$$

$$s(p_n, \sigma_1) = \frac{n p_n (1 + \sigma p_n)^{3/2} \sqrt{p_n(1 + p_n)}}{(p_n^2 + \sigma_1^2)\{(1 + 2p_n)\sqrt{1 + \sigma p_n} - \sigma\sqrt{p_n(1 + p_n)}} \qquad \ldots(21)$$

and for $s(p_n, \sigma_2)$ write $\sigma_2$ in place of $\sigma_1$ in $s(p_n, \sigma_1)$.

Also, from eqn. (13) (the case of double poles), the skin friction on the disks is given by

$$\tau_{\omega(\bar{y}=0)} = \bar{\mu}\Big[A_1\{1 - (1-\sigma) i\sigma_1\} e^{i\sigma_1 \bar{t}} s(\sigma_1) \coth s(\sigma_1) \bar{h}$$

$$+ B_1\{1 + (1-\sigma) i\sigma_1\} e^{-i\sigma_1 t} s(-\sigma_1) \coth s(-\sigma_1) \bar{h} +$$

(*equation continued on p.* 488)

$$+ C_1 \{1 - (1 - \sigma)\, i\sigma_2\}\, e^{i\sigma_2 \bar{t}}\, s(\sigma_2) \operatorname{cosech} s(\sigma_2)\, \bar{h}$$

$$+ D_1 \{1 + (1 - \sigma)\, i\sigma_2\}\, e^{-i\sigma_2 \bar{t}}\, s(-\sigma_2) \operatorname{cosech} s(-\sigma_2)\, \bar{h}$$

$$+ \sum_{n=0}^{\infty} L_1 e^{-n\pi \bar{t}/\bar{h}} + \frac{(1-\sigma)}{\bar{h}} \sum_{n=0}^{\infty} L_1 n\pi\, e^{-n\pi \bar{t}/\bar{h}}$$

$$+ \sum_{n=0}^{\infty} \bar{t} M_1\, e^{-n\pi \bar{t}/\bar{h}} + \frac{(1-\sigma)\,\pi}{\bar{h}} \sum_{n=0}^{\infty} n M_1 \bar{t}\, e^{-n\pi \bar{t}/\bar{h}}$$

$$- (1-\sigma) \sum_{n=0}^{\infty} M_1\, e^{-n\pi \bar{t}/\bar{h}} \Bigg] \qquad \ldots(22)$$

$$\tau_{\omega(\bar{y}=\bar{h})} = \bar{\mu} \Bigg[ A_1 \{1 - (1 - \sigma)\, i\sigma_1\}\, e^{i\sigma_1 \bar{t}}\, s(\sigma_1) \operatorname{cosech} s(\sigma_1)\, \bar{h}$$

$$+ B_1 \{1 + (1 - \sigma)\, i\sigma_1\}\, e^{-i\sigma_1 \bar{t}}\, s(-\sigma_1) \operatorname{cosech} s(-\sigma_1)\, \bar{h}$$

$$+ C_1 \{1 - (1 - \sigma)\, i\sigma_2\}\, e^{i\sigma_2 \bar{t}}\, s(\sigma_2) \coth s(\sigma_2)\, \bar{h}$$

$$+ D_1 \{1 + (1 - \sigma)\, i\sigma_2\}\, e^{-i\sigma_2 \bar{t}}\, s(-\sigma_2) \coth s(-\sigma_2)\, \bar{h}$$

$$+ \sum_{n=0}^{\infty} L_2\, e^{-n\pi \bar{t}/\bar{h}} + \frac{(1-\sigma)\,\pi}{\bar{h}} \sum_{n=0}^{\infty} n L_2\, e^{-n\pi \bar{t}/\bar{h}}$$

$$+ \bar{t} \sum_{n=0}^{\infty} M_2\, e^{-n\pi \bar{t}/\bar{h}} + \frac{(1-\sigma)\,\bar{t}\pi}{\bar{h}} \sum_{n=0}^{\infty} n M_2 e^{-n\pi \bar{t}/\bar{h}}$$

$$- (1-\sigma) \sum_{n=0}^{\infty} M_1\, e^{-n\pi \bar{t}/\bar{h}} \Bigg] \qquad \ldots(23)$$

where

$$L_1 = \Bigg[ \frac{d}{dp} \Bigg\{ \Bigg( \frac{C_1}{p - i\sigma_2} + \frac{D_1}{p + i\sigma_2} \Bigg) k \operatorname{cosech} k\bar{h} - \Bigg( \frac{A_1}{p - i\sigma_1} + \frac{B_1}{p + i\sigma_1} \Bigg) \times k \coth k\bar{h} \Bigg\} \Bigg( p + \frac{n\pi}{\bar{h}} \Bigg)^2 \Bigg]_{p=-n\pi/\bar{h}}$$

$$M_1 = \Bigg[ \Bigg\{ \Bigg( \frac{C_1}{p - i\sigma_2} + \frac{D_1}{p + i\sigma_2} \Bigg) k \operatorname{cosech} k\bar{h} - \Bigg( \frac{A_1}{p - i\sigma_1} + \frac{B_1}{p + i\sigma_1} \Bigg) \times k \coth k\bar{h} \Bigg\} \Bigg( p + \frac{n\pi}{\bar{h}} \Bigg)^2 \Bigg]_{p=-n\pi/\bar{h}}$$

$$L_2 = \left[\frac{d}{dp}\left\{\left(\frac{C_1}{p - i\sigma_2} + \frac{D_1}{p + i\sigma_2}\right) k \coth k\bar{h} - \left(\frac{A_1}{p - i\sigma_1} + \frac{B_1}{p + i\sigma_1}\right) \times k \operatorname{cosech} k\bar{h}\right\}\left(p + \frac{n\pi}{\bar{h}}\right)^2\right]_{p=-n\pi/\bar{h}}$$

$$M_2 = \left[\left\{\left(\frac{C_1}{p - i\sigma_2} + \frac{D_1}{p + i\sigma_2}\right) k \coth k\bar{h} - \left(\frac{A_1}{p - i\sigma_1} + \frac{B_1}{p + i\sigma_1}\right) \times k \operatorname{cosech} k\bar{h}\right\}\left(p + \frac{n\pi}{\bar{h}}\right)^2\right]_{p=-n\pi/\bar{h}}.$$

## 5. Rate of Energy Dissipation

Rate of energy dissipation from the oscillating disks can be found from the mean values

$$-\overline{(\tau_{\omega}\bar{u})}_{\bar{y}=0} \text{ and } -\overline{(\tau_{\omega}\bar{u})}_{\bar{y}=\bar{h}}$$

## 6. Discussion

On taking the motion considerably long time after the oscillations of the disks, the transients have the time to die down. The damping of the oscillations as one moves perpendicular to the disks is smaller in viscoelastic fluid in comparison to a fluid possessing the same viscosity but no elasticity and the rate of change of phase is quicker in viscoelastic fluid. The effect of elasticity in the flow pattern of a fluid between disks is to produce further disturbances partly of the same and partly of higher harmonic.

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