ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF PERTURBED DIFFERENCE EQUATIONS

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The objective of this paper is to investigate sufficient conditions for asymptotic stability problem of a nonlinear perturbed system of difference equations assuming the exponential asymptotic stability of the corresponding linear system. The variation of constants formula used here is similar to the usual formula for ordinary and functional differential equations.

§1. Recently, Coffman (1964), Pachpatte (1971), Sugiyama (1969) among others, have discussed the asymptotic stability properties of solutions of perturbed difference equations and obtained several interesting results. Halanay (1963) has studied perturbation problems for difference equations by adopting some constructions used by Massera for differential equations and presented necessary and sufficient conditions, in terms of Lyapunov functions for uniform stability and uniform asymptotic stability. This approach has two natural drawbacks, namely, that no general techniques are available for finding suitable Lyapunov function for the unperturbed system, and use it as a suitable Lyapunov function for the perturbed system ; and moreover it is rather difficult to obtain quantitative estimates by this method.

In the present paper we shall show that under same suitable conditions the exponential stability of linear system implies the asymptotic stability of nonlinear perturbed system of difference equations. The variation of constants formula used here is similar to the usual formula for ordinary and functional differential equations.

§2. We consider the system of perturbed difference equations

$$\text{(D)}\quad Ex(t) = A(t) + f(t, x(t)) + g(t, x(t))$$

where each x, f and g are elements of R^n, an n-dimensional vector space, $A(t)$ is an $n \times n$ matrix with $\det(A(t)) \neq 0$, E is an operator such that $Ex(t) = x(t+1)$ and the functions $f(t, x)$ and $g(t, x)$ are defined on $I \times G \to R^n$, where I denotes the integers $0 \leqslant t < \infty$, and G be an open set in R^n. For any n vector $x \in R^n$. Let $\| x \|$ denote any appropriate vector norm of x. We denote by $x(t) = x(t; t_0, x_0)$ the solution of (D) with the initial condition $x(t_0; t_0, x_0) = x_0$, $t_0 \geqslant 0$. We suppose that $f(t, 0) = 0$, in $t \in I$. Let $g(t, x)$ satisfy H_1 : there exists $d > 0$, such that if $\| x \| \leqslant d$, then $\| g(t, x) \| \leqslant r(t)$ for all $t \geqslant 0$, where $r(t) \to 0$, as $t \to \infty$.

Lemma 1 (Pachpatte 1973, p. 354) — The linear difference system

$$(D') \; Ex(t) = A(t)\, x(t)$$

is exponentially asymptotically stable if and only if there exist constants $\beta \geqslant 1$ and $\alpha > 0$ such that

$$\| X(t)\, X^{-1}(s) \| \leqslant \beta\, e^{-\alpha(t-s)} \text{ for all } t \geqslant s \geqslant 0.$$

where $X(t)$ denotes a fundamental matrix of (D'). We require the following lemma in our subsequent discussion.

Lemma 2 — Let the functions $u(t)$ and $r(t)$ be non-negative and defined for $t \geqslant t_0$, let $c \geqslant 0, k \geqslant 0$, and if

$$u(t) \leqslant c + \sum_{s=t_0}^{t-1} [ku(s) + r(s)]$$

then

$$u(t) \leqslant c \,.\, e^{k(t-t_0)} + \sum_{s=t_0}^{t-1} r(s)\, e^{k(t-s-1)}.$$

PROOF : Let

$$m(t) = c + \sum_{s=t_0}^{t-1} [ku(s) + r(s)],\; m(t_0) = c.$$

Then we obtain a relation

$$m(t+1) \leqslant (1+k)\, m(t) + r(t).$$

The result follows by substituting successively $t = t_0, t_0 + 1, \ldots, t-1$ in the above relation.

§3. We discuss here the main result.

Theorem 1 — Let (D') be exponentially asymptotically stable, with corresponding constants β and α. Suppose that for sufficiently small constant $c > 0$, $f(t, x)$ satisfies the inequality, $\| f(t, x) \| \leqslant c \| x \|$ for all $t \in I$ and $\| x \| < h$, and $g(t, x)$ satisfies (H_1). Then all the solutions of (D) approach zero as $t \to \infty$.

PROOF : Using the variation of constant formula similar to that of differential equations, any solution $y(t)$ of (D) can be represented (cf. Sugiyama 1969, p. 141) by

$$y(t) = X(t)\, X^{-1}(t_0)\, x_0 + \sum_{s=t_0}^{t-1} X(t)\, X^{-1}(s+1)\, [f(s, x(s)) + g(s, x(s))]$$

which in view of the exponential stability of (D′) implies

$$\| x(t) \| \leqslant \beta e^{-\alpha(t-t_0)} \| x_0 \| + \sum_{s=t_0}^{t-1} \beta e^{-\alpha(t-s-1)} [c \| x(s) \| + r(s)].$$

The above inequality can be rewritten as

$$\| x(t) \| e^{\alpha t} \leqslant \beta e^{\alpha t_0} \| x_0 \| + \sum_{s=t_0}^{t-1} [c\beta e^{\alpha} \| x(s) \| e^{\alpha s} + \beta e^{\alpha(s+1)} r(s)].$$

Now applying Lemma 2 with $u(t) = \| x(t) \| e^{\alpha t}$ yields

$$\| x(t) \| \leqslant \beta \| x_0 \| e^{-(\alpha - c\beta e^{\alpha})(t-t_0)} + \sum_{s=t_0}^{t-1} \beta e^{-(\alpha - c\beta e^{\alpha})(t-s-1)} r(s).$$

Thus if we choose c and $\| x_0 \|$ so small that $\alpha - c\beta e^{\alpha} > 0$ and $\beta \| x_0 \| < h$ and since the above inequality is satisfied for any $t \in I$, it follows that $\| x(t) \| \to 0$ as $t \to \infty$. This completes the proof.

Remark : Theorem 1 is a type of asymptotic stability theorem for the system (D).

Example : Consider the equations

(L) $Ex(t) = A(t)\, x(t)$

(L′) $Ey(t) = [A(t) + B(t)]\, y(t)$

where $A(t)$ is an $n \times n$ matrix with det $(A(t)) \neq 0$ and $\| B(t) \| \leqslant c$. Suppose that the solution of (L) is exponentially asymptotically stable with corresponding constants β and α, then for any solution $y(t)$ of (L′) such that $\| y(t) \| \leqslant h$ the inequality

$$\| y(t) \| \leqslant \beta e^{-\alpha(t-t_0)} \| y_0 \| + \sum_{s=t_0}^{t-1} \beta e^{-\alpha(t-s-1)}\, c \| y(t) \|$$

is satisfied. This implies that

$$\| y(t) \| \leqslant \beta \| y_0 \| \exp\,(-(\alpha - c\beta e^{\alpha})(t - t_0)).$$

Hence if we choose c and $\| x_0 \|$ so small that $\alpha - c\beta e^{\alpha} > 0$ and $\beta \| y_0 \| < h$ are satisfied, then $\| y(t) \| \to 0$ as $t \to \infty$.

Remark : Note that the above results include, in particular, the asymptotic stability of the zero solution of the perturbed difference equations (D) when $g(t, x) \equiv 0$.

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