SPECTRALOID OPERATORS, SIMILARITY AND RELATED RESULTS

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We are mainly concerned here with similarity of operators involving their inverses and adjoints. We extend a recent result of DePrima (1974) and a result of Sheth (1969) to spectraloid operators; these considerations lead us to answer Istratescu's conjecture (1970) in the affirmative.

We also prove certain results for invertible operators in connection with cosquare that help us to solve one way implication of a conjecture by Choi (1973). A partial answer to a conjecture of Radjavi and Williams (1969) is supplied. Generalizations of some results of Patel (1973) are accomplished.

1. INTRODUCTION

Let $\beta(H)$ denote the Banach algebra of all bounded linear operators on a Hilbert space H. Denote by $\sigma(T)$ the spectrum of $T \in \beta(H)$ and by $W(T)$ the numerical range of T; $r(T)$, $w(T)$ denote respectively the spectral radius and numerical radius of T. An operator T is called normaloid if $W(T) = \| T \|$; convexoid if $\overline{W(T)} = \text{Conv.}\ \sigma(T)$. (Here, $\overline{W(T)}$ denotes the closure of $W(T)$ and conv. denotes the convex hull); spectraloid if $r(T) = w(T)$. A normaloid operator need not be convexoid nor a convexoid operator be a normaloid; however, convexoid operators and normaloid operators are both spectraloid operators (see Halmos 1967).

In what follows $\triangle$ denotes the closed unit disc with centre at the origin in the complex plane.

In this paper we are primarily concerned with similarity of operators involving their inverses or adjoints. We shall postpone detailed discussion to respective sections.

2. SPECTRALOID OPERATORS AND SIMILARITY

In this section we shall be interested in the conditions on spectraloid operators implying normality or uniticity. Our study leads us to generalizations of some of the results of DePrima (1974), Sheth (1969), Patel (1973) and Williams (1969). As our first result we have :

Theorem 2.1 — Let $T, S \in \beta(H)$ with T invertible. If

(i) $T^{-1}S = ST^*$ with $0 \notin \overline{W(S)}$ and

(ii) both T and T^{-1} are spectraloid, then T is unitary.

PROOF : On account of hypothesis (i), we see that T is similar to a unitary operator; see Singh and Kanta Mangla (1973). Hence $r(T) = 1$. If in addition T is spectraloid we then have $r(T) = w(T) = 1$, and hence $W(T) \subset \triangle$, the unit disc in the complex plane. Since T^{-1} is also similar to a unitary operator we have $r(T^{-1}) = 1$ and as T^{-1} is spectraloid we infer that $r(T^{-1}) = w(T^{-1}) = 1$ and thus again $W(T^{-1}) \subset \triangle$. As $W(T^{\pm 1}) \subset \triangle$, we get on account of a result of Stampfli (1967, Cor. 1, p 605) that T is unitary.

Q.E.D.

As particular cases we obtain the results of DePrima (1974) and Singh and Kanta Mangla (1973).

If the underlying Hilbert space is finite dimensional then the hypothesis that T^{-1} is also spectraloid can be dropped. In fact, we have the following :

Theorem 2.2 — Let H be a finite-dimensional Hilbert space. If $S, T \in \beta(H)$ with T invertible are such that

(i) $T^{-1}S = ST^*$ with $0 \notin \overline{W(S)}$ and

(ii) T is spectraloid, then T is unitary.

PROOF : As was done in the infinite dimensional case (i) of the hypothesis implies that T is similar to a unitary operator. Hence T is diagonizable and $\sigma(T) = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ lies in the unit circle $\triangle$. Because T is spectraloid $w(T) = r(T) = 1$ gives that $W(T) \subset \triangle$. Also α_j lie on the boundary of the numerical range $W(T)$. Now $ker(T - \alpha_j) = ker(T^* - \alpha_j)$, $j = 1, 2, \ldots, k$ yield that these subspaces are mutually orthogonal and since T is diagonizable, they reduce T and span H. So T is unitarily equivalent to a diagonal matrix and thus is unitary.

Q.E.D.

It is observed that this extends a result of DePrima and Johnson (1974), wherein they prove it for normaloid or convexoid operator; see also DePrima (1974).

In order to prove our next result we need the following result due to Furuta and Nakamoto (1971).

Lemma (Furuta-Nakamoto) — An operator T is convexoid if and only if $T - \alpha$ is spectraloid for every complex scalar α.

Theorem 2.3 — Let T be an operator such that $T - \alpha$ is spectraloid for every complex scalar α. If $ST = T^*S$ for an operator S with $0 \notin \overline{W(S)}$, then T is self-adjoint.

PROOF : By Furuta-Nakamoto lemma T is clearly a convexoid operator. If in addition $ST = T^*S$ for an operator S with $0 \notin \overline{W(S)}$ then in view of Williams' result (1969) $\sigma(T)$ is real and convexoid nature of T implies that $\overline{W(T)}$ is real. Hence we conclude that T is self-adjoint.

Q.E.D.

Similar result was proved by Sheth (1969) for $T - \alpha$ to be normaloid for every α. Istratescu (1970) proves the following :

Theorem — If Σ is a subset of the complex plane, and $T - \alpha$ is normaloid for all $\alpha \in \Sigma$, then T is normal,

(i) trivially if dim. $H = 1$,

(ii) if dim. $H = 2$ and Σ is any point,

(iii) if dim. $H = 3$ and Σ is any set of two distinct points, and

(iv) if dim. $H = 4$ and Σ is any set of three distinct points.

He further raises an open problem whether this theorem is valid for $T - \alpha$ spectraloid for every α. We remark that our Theorem 2.3 not only answers this problem but proves much more.

In the same vein one can also prove the following :

Theorem 2.4 — If $T - \alpha$ is spectraloid for every complex scalar α and if $T^{-1}S = ST^*$ with $0 \notin \overline{W(S)}$, then T is unitary if $w(T) = w(T^{-1})$.

Lastly, we are concerned in this section with considerations which stem from the following well-known problem of Halmos (1967, problem 165) :

"Is a contraction similar to a unitary operator necessarily unitary?"

We raise a similar problem for spectraloid operators.

We ask : Is a spectraloid operator similar to a unitary operator necessarily unitary? As an answer we are led to the following result the proof of which is easy.

Theorem 2.5 — If T is an invertible spectraloid operator which is similar to a unitary operator and T^{-1} is a numerical contraction, then T is unitary.

Khasbardar and Thakare (1977) recently introduced and investigated several concepts which are in a sense generalizations of the concept of similarity. An operator T is called isometrically equivalent to an operator S if there exists an isometry M such that $T = MSM^*$. Khasbardar and Thakare (1977) also show that under isometric equivalence spectraloid operators remain invariant. The following result also throws some light on the problem that we just raised.

Theorem 2.6 — Let T be a spectraloid operator similar to a unitary operator U. If T^* is isometrically equivalent to T^{-1} then T is unitary.

PROOF: The assumption $T = S^{-1}US$ implies that $T^{-1} = S^{-1}U^*S$. Since spectrum is preserved under similarity, we have $r(T) = r(U) = 1$ and $r(T^{-1}) = r(U^*) = 1$. Again T^* is isometrically equivalent to T^{-1} implies $w(T^{-1}) \leqslant w(T) = r(T) = 1$. Thus $W(T^{\pm 1}) \subset \triangle$; and by Stampfli's result (1967) we infer that T is unitary.

Q.E.D.

3. INVERTIBLE OPERATORS AND CONJECTURES ON SIMILARITY

Choi (1973) introduced the concept of cosquare for invertible operators. For an invertible operator T, T^{*-1} is called the cosquare of T. In the course of his investigations Choi (1973) raised the following problem.

Choi's Conjecture

Let T be invertible. Then T^* is similar to T^{-1} if and only if T is a cosquare.

If T is a cosquare, then by definition there exists an invertible operator S satisfying $T = S^{*-1}S$. Then $T^* = S^*S^{-1}$ and $T^{-1} = S^{-1}S^*$ show that $T^{-1} = S^{-1}S^*S^{-1}S = S^{-1}T^*S$ and so T^* is similar to T^{-1}. Thus one way implication of Choi's conjecture is obvious. In view of this the said conjecture can be formulated as: "An operator T is a cosquare if T^* is similar to T^{-1}". If T is a cosquare of itself then the conjecture is solved in the affirmative. In fact, in this case $T = I$ and every invertible operator gives the desired similarity.

Choi (1973) could characterize invertible normal operators. He showed that "an invertible operator T is normal if and only if cosquare of T is unitary". This motivates us to obtain a similar characterization of a hyponormal operator. In fact, we have

Theorem 3.1 — An invertible operator T is hyponormal if and only if cosquare of T is a contraction.

PROOF: Hyponormality of T implies that

$$(T^{*-1}T)\,(T^{*-1}T)^* = T^{*-1}TT^*T^{-1} \leqslant T^{*-1}T^*TT^{-1} = T.$$

Hence $T^{*-1}T$ is a contraction. Conversely, if $T^{*-1}T$ is a contraction then T is hyponormal.

Q.E.D.

Choi's concept of conormal operators suggests that one can introduce the following notion. An operator T is conormal if T^2 is unitary.

Definition — An invertible operator T is called hypoconormal if

$$(T^{*-1}T)\,(T^{*-1}T)^* \leqslant (T^{*-1}T^{-1})\,(T^{*-1}T^{-1})^*.$$

Hypoconormal operators are characterized in the following

Theorem 3.2 — An invertible operator T is hypoconormal if and only if $T^*T \leqslant (T^*T)^{-1}$.

PROOF : From the definition $(T^{*-1}T)(T^{*-1}T)^* \leqslant (T^{*-1}T^{-1})(T^{*-1}T^{-1})^*$ if and only if $T^{*-1}TT^*T^{-1} \leqslant T^{*-1}T^{-1}T^{*-1}T^{-1}$ if and only if $TT^* \leqslant T^{-1}T^{*-1} = (T^*T)^{-1}$.

Q.E.D.

Even though this result is easy to prove it leads to an additional characterization of conormal operators. In fact, one can prove that T is conormal if and only if $(T^{*-1}T)(T^{*-1}T)^* = (T^{*-1}T^{-1})(T^{*-1}T^{-1})^*$. Also Theorem 3.2 yields that T is hypoconormal if and only if T^2 is a contraction. One also notes that cosquare of an operator T is unitary if and only if T commutes with its cosquare; and T is normal if and only if T commutes with its cosquare.

To close this section, our interest will be in the following equivalent formulation of the conjecture of Radjavi and Williams (1969).

Conjecture (Radjavi-Williams 1969)

If T is similar to T^*, then some similarity of T is unitarily equivalent to its adjoint.

Since two similar normal operators are unitarily equivalent and adjoint of a normal operator is again normal, the above conjecture is known to be solved for normal operators.

We furnish a partial answer of the Radjavi-Williams conjecture in the form of

Theorem 3.3 — Let T be similar to T^* via a normal operator S. Then some similarity of T is unitarily equivalent to its adjoint.

PROOF : Let $T^* = STS^{-1}$ with S normal; and let $S = UP$ be its polar decomposition. Then U and P commute and $T^* = UPTP^{-1}U^* = PUTU^*P^{-1}$.

Now let R be the positive square root of P, then $R^2 = P$ and R also commutes with U. Hence $T^* = R^2UTU^*R^{-2}$ or $R^{-1}T^*R = RUTU^*R^{-1}$ or $R^{-1}T^*R = URTR^{-1}U^*$. Thus RTR^{-1}, the similarity of T is unitarily equivalent to its adjoint $R^{-1}T^*R$. This completes the proof.

Q.E.D.

4. Additional Results

Patel (1973) obtains several results involving similarity and left inverses. Here are some variants that stem from his study.

Theorem 4.1 — Let T be an invertible operator. If there exists a normal operator S with polar decomposition UP and $T^* = S^{-1}T^{-r}S$ for $0 \notin \overline{W(S)}$ and r a positive integer, then T is similar to a unitary operator if U commutes with T.

PROOF: We have $T^* = P^{-1}U^*T^{-r}UP$. By hypothesis P is positive and invertible; and $P^{1/2}$ will be its positive square root. Let $A = P^{-1/2}T P^{1/2}$ and $B = P^{-1/2}T^{-1}P^{1/2}$. Then $A^* = P^{1/2}T^*P^{-1/2}$ and

$$B^r = P^{-1/2}T^{-r}P^{1/2} = P^{-1/2}P\ UT^*P^{-1}P^{1/2} = P^{1/2}T^*P^{-1/2} = A^*.$$

Also $A^{-r} = P^{-1/2}T^{-r}\ P^{-1/2} = A^*$; thus A is normal. Hence $\|A\| = \|A^{-r}\| = \|A^{-1}\|^r$. Again using a result of Patel (1973), $\sigma(T)$ lies on the unit disc. We thus have $\|A\| = \|A^{-1}\| = 1$; hence A is unitary.

Q.E.D.

Theorem 4.2 — Let T be an invertible normal operator and let $T^* = S^{-1}T^{-k}S$ with $0 \notin \overline{W(S)}$ and $k \geqslant 2$. Then T^{k-1} is a projection.

PROOF: It is easy to see that $T^* = S^{-1}T^{-k}S$ implies T is unitary. Next, $ST^* = T^{-k}S$ and commuting nature of T^*, T^{-k} gives that $T^* = T^{-k}$; see Embry (1970). Uniticity of T yields $T^{-1} = T^*$ and $T^* = T^{*k}$, so that $T = T^k$. Hence T^{k-1} is a projection on account of Furuta and Nakamoto (1969).

Q.E.D.

By adopting similarity arguments Williams (1969) and Patel (1973) obtain certain results regarding the spectrum of a given operator. By reproducing the arguments of Williams (1969) one can obtain the following result in which the condition $0 \notin \overline{W(S)}$ is replaced by $0 \notin W(S)$.

Result (Williams 1969)

If T is an operator with $S^{-1}TS = T^*$ and $0 \notin W(S)$, then the set of eigenvalues of T is real.

Lastly, we shall have a generalization of a result of Patel (1973).

Theorem 4.3 — Let T be left invertible operator with left inverse T_1. Let $T^* = S^{-1}T_1^k\ S$ and $0 \notin W(S)$, k a non-negative integer, then $\sigma_p(T)$ lies in the unit circle.

PROOF: Let $\alpha \in \sigma_p(T)$, then $Tx = x$ for some $x \in H$. T_1 being left inverse of T, we have $T_1x = \alpha^{-1}x$. Starting with $|\ (\alpha^{-k} - \alpha^*)\ (S_x^{-1}, x)\ |$ we see that it is less than or equal to zero from which we get $\alpha = \alpha^{*k}$ or $|\ \alpha\ | = 1$.

Q.E.D.

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