ON NÖRLUND SUMMABILITY OF JACOBI SERIES

by G. S. Pandey *and* B. K. Beohar, *Department of Mathematics, Vikram University, Ujjain*

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In this paper, the authors have studied the Nörlund summability of Jacobi series under the conditions which improve all the previously known theorems in this line of work.

1. Definitions

Let Σu_n be a given infinite series with a sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex.

Let us write

$$P_n = p_0 + p_1 + p_2 + \dots p_n.$$

The sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_{n-\nu} = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu \qquad \dots(1.1)$$

defines the sequence $\{t_n\}$ of the Nörlund mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series is said to be summable (N, p_n) to the sum s if the limit t_n exists and is equal to s.

Two important particular cases of (N, p_n) summability are :

(i) Harmonic summability, when $p_n = \dfrac{1}{n+1}$;

(ii) Cesaro summability, when $p_n = \dbinom{n+\delta-1}{\delta-1}$, $\delta > 0$.

2. Introduction

Let $f(x)$ be defined in the closed interval $[-1, 1]$ such that the function

$$(1-x)^\alpha (1+x)^\beta f(x) \in L[-1, 1], \ \alpha > -1, \ \beta > -1.$$

The Jacobi series corresponding to this function is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \qquad \dots(2.1)$$

where

$$a_n = \frac{(2n+\alpha+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)} \times \int_{-1}^{1} (1-x)^{\alpha}\,(1+x)^{\beta}\,f(x)\,P_n^{(\alpha,\beta)}(x)\,dx \qquad \ldots(2.2)$$

and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials.

We write

$$F(\phi) \equiv \{f(\cos\phi) - A\} \left(\sin\frac{\phi}{2}\right)^{2\alpha+1} \left(\cos\frac{\phi}{2}\right)^{2\beta+1}$$

A being a fixed constant.

Dealing with (N, p_n) summability of Jacobi series Gupta (1970) proved the following :

Theorem A — Let $\{p_n\}$ be a non-negative and non-increasing sequence such that

$$\sum_{k=a}^{n} \frac{P_k}{k^{(2\alpha+3)/2}\log k} = O\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right) \qquad \ldots(2.3)$$

a being a fixed positive integer and

$$\sum_{n} \frac{n^{(2\alpha+1)/2}}{P_n} < \infty. \qquad \ldots(2.4)$$

If

$$\psi(t) \equiv \int_0^t |F(\phi)|\,d\phi = o\left(\frac{t^{2\alpha+2}}{\log 1/t}\right) \text{ as } t \to 0 \qquad \ldots(2.5)$$

then the series (2.1) is summable (N, p_n) at $x = 1$ to the sum A provided $-\frac{1}{2} \leqslant \alpha < \frac{1}{2}$; $\beta > -\frac{1}{2}$ and the antipole condition,

$$\int_{-1}^{b} (1+x)^{(2\beta-3)/4}\,|f(x)|\,dx < \infty \qquad \ldots(2.6)$$

b fixed, is satisfied.

Recently Choudhary (1972) has established this result under the condition which is weaker than (2.5) and without employing (2.3). In fact, he has established the following :

Theorem B — Let $\{p_n\}$ be a real non-increasing, non-negative sequence of coefficients. If

$$\psi(t) = o\left(\frac{p(1/t)\, t^{2\alpha+1}}{P(1/t)}\right) \text{ as } t \to 0 \qquad \text{...(2.7)}$$

(2.4) and (2.6) also hold then the series (2.1) is summable (N, p_n) at $x = 1$ to the sum A for $-\frac{1}{2} \leqslant \alpha < \frac{1}{2}$; $\beta > -\frac{1}{2}$.

Looking into the proof of this theorem we observe that he has used an additional condition

$$\sum_{a}^{n} \frac{P_k}{k^{(2\alpha+3)/2}} = O\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right), \; P(n) = P_n \qquad \text{...(2.8)}$$

without mentioning it in the statement of the theorem.

In this paper we have been able to show that the result holds even if (2.7) and (2.8) are replaced by a still weaker condition. Precisely, we shall prove the following theorem:

Theorem — Let $\{p_n\}$ be a positive, non-increasing sequence of real numbers such that $\{n^{-(2\alpha+1)/2} P_n\}$ is increasing,

$$\int_{t}^{\delta} \frac{|F(\phi)|\, P\left(\frac{1}{\phi}\right)}{\phi^{(2\alpha+3)/2}}\, d\phi = o\left\{t^{(2\alpha+1)/2} P\left(\frac{1}{t}\right)\right\} \qquad \text{...(2.9)}$$
$$t \to 0$$

and (2.6) also holds, then the series (2.1) is summable (N, p_n) at $x = 1$ to the sum A for $-\frac{1}{2} \leqslant \alpha < \frac{1}{2}$, $\beta > -\frac{1}{2}$.

It may be observed (as we shall show in Lemma 5) that the condition (2.7) together with (2.8) imply (2.9).

3. Lemmas

We require the following lemmas for the proof of the theorem.

Lemma 1 — For α, β arbitrary and real and C a fixed positive constant,

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-(2\alpha+1)/2}\, O(n^{-1/2}) \text{ for } \dfrac{C}{n} \leqslant \theta \leqslant \dfrac{\pi}{2} \\ O(n^{\alpha}) \text{ for } 0 \leqslant \theta \leqslant \dfrac{C}{n}. \end{cases} \qquad \text{...(3.1)}$$

[For the proof see Szegö (1959, p. 167).]

Lemma 2 — If $\alpha > -1$, $\beta > -1$; $\dfrac{C}{n} \leqslant \theta \leqslant \pi - \dfrac{C}{n}$,

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2}\, u(\theta) \left[\cos\left\{ \left(n + \frac{\alpha+\beta+1}{2} \right)\theta + \gamma \right\} + \frac{O(1)}{n \sin\theta} \right] \qquad \ldots(3.2)$$

where

$$u(\theta) = \frac{1}{\sqrt{\pi}} \left(\sin\frac{\theta}{2} \right)^{-(2\alpha+1)/2} \left(\cos\frac{\theta}{2} \right)^{-(2\beta+1)/2}; \quad \gamma = -\left(\alpha + \tfrac{1}{2}\right)\frac{\pi}{2}\cdot$$

[See Szegö (1959, p. 196)].

Lemma 3 — Let

$$N_n(\phi) = \frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{n} p_k \lambda_{n-k}\, P_{n-k}^{(\alpha+1,\beta)}(\cos\phi)$$

where

$$\lambda_n = \frac{2^{-\alpha-\beta-1}\,\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} \cong \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)}\, n^{\alpha+1}.$$

For $0 \leqslant \phi \leqslant \dfrac{1}{n}$,

$$N_n(\phi) = O(n^{2\alpha+2}). \qquad \ldots(3.3)$$

For $\dfrac{1}{n} \leqslant \phi \leqslant \pi - \dfrac{1}{n}$,

$$N_n(\phi) = \frac{1}{P_n}\, O\left[n^{(2\alpha+1)/2}\, P\left(\frac{1}{\phi}\right) \Big/ \left(\sin\frac{\phi}{2} \right)^{(2\alpha+3)/2} \left(\cos\frac{\phi}{2} \right)^{(2\beta+1)/2} \right.$$

$$\left. + O\left[n^{(2\alpha-1)/2} \Big/ \left(\sin\frac{\phi}{2} \right)^{(2\alpha+5)/2} \left(\cos\frac{\phi}{2} \right)^{(2\beta+3)/2} \right] \right]. \qquad \ldots(3.4)$$

PROOF : By using the order estimates given in (3.1) we can easily establish (3.3) while (3.4) can be obtained by an application of (3.2).

Lemma 4 — The antipole condition (2.6) implies that

$$\int_a^\pi \left(\cos\frac{\phi}{2} \right)^{(2\beta-1)/2} | f(\cos\phi) - A \,|\, d\phi < \infty \qquad \ldots(3.5)$$

which further implies that

$$\int_0^{1/n} t^{(2\beta-1)/2}\, | f(-\cos t) - A \,|\, dt = o(1). \qquad \ldots(3.6)$$

[See Gupta (1970) for its proof.]

Lemma 5 — Conditions (2.7) and (2.8) imply (2.9).

PROOF : We have

$$\int_t^\delta \frac{|F(\phi)|\, P\left(\frac{1}{\phi}\right)}{\phi^{(2\alpha+3)/2}}\, d\phi = \left\{ o\left(\frac{p\left(\frac{1}{\phi}\right)\phi^{2\alpha+1}}{P\left(\frac{1}{\phi}\right)} \right) \frac{P\left(\frac{1}{\phi}\right)}{\phi^{(2\alpha+3)/2}} \right\}_t^\delta$$

$$+ o\int_t^\delta \frac{p\left(\frac{1}{\phi}\right)\phi^{2\alpha+1}}{P\left(\frac{1}{\phi}\right)} \left| \frac{d}{d\phi}\left(\frac{P\left(\frac{1}{\phi}\right)}{\phi^{(2\alpha+3)/2}} \right) \right| d\phi$$

$$= o\left(t^{(2\alpha+1)/2}\, P\left(\frac{1}{t}\right)\right) + o\int_{1/\delta}^{n} \frac{p(x)}{P(x)\, x^{2\alpha+1}} \left| \frac{d}{dx}\{P(x)\, x^{(2\alpha+3)/2}\} \right| dx$$

$$= o\left(t^{(2\alpha+1)/2}\, P\left(\frac{1}{t}\right)\right) + \sum_{a}^{n} \frac{p(k)}{P(k)\, k^{2\alpha+1}} \mid \Delta\{P(k)\, k^{(2\alpha+3)/2}\} \mid$$

$$+ O(1), \quad a = \left[\frac{1}{\delta}\right] + 1, \quad n \geqslant \left[\frac{1}{t}\right].$$

$$= o\left(t^{(2\alpha+1)/2}\, P\left(\frac{1}{t}\right)\right) + o\sum_{a}^{n} \frac{P(k)}{k^{(2\alpha+3)/2}}$$

$$= o\left(t^{(2\alpha+1)/2}\, P\left(\frac{1}{t}\right)\right) + o\left(\frac{P(n)}{n^{(2\alpha+1)/2}}\right), \quad \text{by (2.8)}.$$

Lemma 6 — The condition (2.9) implies that

$$\int_0^t |F(\phi)|\, d\phi = o(t^{2\alpha+2}). \qquad \ldots(3.7)$$

PROOF : We have

$$\eta(t) \equiv \int_t^\delta \frac{|F(\phi)|\, P\left(\frac{1}{\phi}\right)}{\phi^{(2\alpha+3)/2}}\, d\phi = o\left(t^{(2\alpha+1)/2}\, P\left(\frac{1}{t}\right)\right).$$

Therefore,

$$\int_0^t |F(\phi)|\, P\left(\frac{1}{\phi}\right) d\phi = \int_0^t \frac{|F(\phi)|\, P\left(\frac{1}{\phi}\right)}{\phi^{(2\alpha+3)/2}} \cdot \phi^{(2\alpha+3)/2}\, d\phi$$

(*equation continued on p.* 506)

$$= [\phi^{(2\alpha+3)/2}\,\eta(\phi)]_0^t + O\int_0^t \phi^{(2\alpha+1)/2}\,|\,\eta(\phi)\,|\,d\phi$$

[on integration by parts]

$$= \left[\phi^{(2\alpha+3)/2}\, o\left(\phi^{(2\alpha+1)/2}\, P\left(\frac{1}{\phi}\right)\right)\right]_0^t + O\int_0^t \phi^{(2\alpha+1)/2}$$

$$\times\, o\left\{\phi^{(2\alpha+1)/2}\, P\left(\frac{1}{\phi}\right)\right\} d\phi$$

$$= o\left(t^{2\alpha+2}\, P\left(\frac{1}{t}\right)\right) + o\int_0^t \phi^{2\alpha+1}\, P\left(\frac{1}{\phi}\right) d\phi$$

$$= o\left(t^{2\alpha+2}\, P\left(\frac{1}{t}\right)\right);\ \text{ as } \frac{P(u)}{u} \text{ is decreasing.}$$

Also, since

$$\int_0^t |\,F(\phi)\,|\,P\left(\frac{1}{\phi}\right) d\phi \geqslant P\left(\frac{1}{t}\right)\int_0^t |\,F(\phi)\,|\,d\phi$$

we get

$$\int_0^t |\,F(\phi)\,|\,d\phi = o(t^{2\alpha+2}).$$

4. Proof of the Theorem

The nth partial sum of the series (2.1) at the point $x = 1$ is given by Obrechkoff (1936) as

$$s_n(1) = 2^{\alpha+\beta+1}\lambda_n \int_0^\pi \left(\sin\frac{\phi}{2}\right)^{2\alpha+1}\left(\cos\frac{\phi}{2}\right)^{2\beta+1} f(\cos\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi.$$

Consequently,

$$s_n(1) - A = 2^{\alpha+\beta+1}\lambda_n \int_0^\pi \left(\sin\frac{\phi}{2}\right)^{2\alpha+1}\left(\cos\frac{\phi}{2}\right)^{2\beta+1} F(\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi \qquad \ldots(4.1)$$

The Nörlund mean $\{t_n\}$ of the series (2.1) at $x = 1$ is given by

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k\, s_{n-k}(1).$$

Therefore,

$$t_n - A = \frac{1}{P_n} \sum_{k=0}^{n} p_k 2^{\alpha+\beta+1} \lambda_{n-k} \int_0^{\pi} F(\phi)\, P_{n-k}^{(\alpha+1,\beta)} (\cos\phi)\, d\phi$$

$$= \int_0^{\pi} F(\phi)\, N_n(\phi)\, d\phi$$

$$= \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-(1/n)} + \int_{\pi-(1/n)}^{\pi}$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say} \qquad \ldots(4.2)$$

δ being a suitable constant.

Making use of the relation (3.1) we get

$$I_1 = O(n^{2\alpha+2}) \int_0^{1/n} | F(\phi) |\, d\phi$$

$$= O(n^{2\alpha+2})\, o(n^{-(2\alpha+2)}), \text{ by (3.7)}$$

$$= o(1). \qquad \ldots(4.3)$$

In order to estimate I_2 we employ the asymptotic relation given in (3.4). Thus

$$I_2 = O \int_{1/n}^{\delta} | F(\phi) | \left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) P\left(\frac{1}{\phi}\right) \left(\sin\frac{\phi}{2}\right)^{-(2\alpha+3)/2} d\phi$$

$$+ O \int_{1/n}^{\delta} | F(\phi) |\, n^{(2\alpha-1)/2} \left(\sin\frac{\phi}{2}\right)^{-(2\alpha+5)/2} d\phi$$

$$= I_{2\cdot 1} + I_{2\cdot 2}, \text{ say.}$$

Now

$$I_{2\cdot 1} = O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) \int_{1/n}^{\delta} \frac{| F(\phi) |\, P\left(\frac{1}{\phi}\right)}{\phi^{(2\alpha+3)/2}}\, d\phi$$

$$= O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) o\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right), \text{ by (2.9)}$$

$$= o(1). \qquad \ldots(4.4)$$

Also,

$$I_{2\cdot 2} = O(n^{(2\alpha-1)/2}) \int_{1/n}^{\delta} \frac{|F(\phi)|}{\phi^{(2\alpha+5)/2}}\, d\phi$$

$$= O(n^{(2\alpha-1)/2}) \left[\left\{ \frac{1}{\phi^{(2\alpha+5)/2}}\; o(\phi^{2\alpha+2}) \right\}_{1/n}^{\delta} + o \int_{1/n}^{\delta} \frac{1}{\phi^{(2\alpha+7)/2}}\, \phi^{2\alpha+2}\, d\phi \right]$$

$$= O(n^{(2\alpha-1)/2}) + o(n^{(2\alpha-1)/2})\,(n^{(1-2\alpha)/2})$$

$$= o(1), \text{ as } \alpha < \tfrac{1}{2}. \qquad \ldots(4.5)$$

Coming to I_3 we have

$$I_3 = O \int_{\delta}^{\pi-(1/n)} \frac{|F(\phi)|\, P\left(\frac{1}{\phi}\right)}{\left(\sin\frac{\phi}{2}\right)^{(2\alpha+3)/2} \left(\cos\frac{\phi}{2}\right)^{(2\beta+1)/2}} \cdot \left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) d\phi$$

$$+ O(n^{(2\alpha-1)/2}) \int_{\delta}^{\pi-(1/n)} \frac{|F(\phi)|}{\left(\sin\frac{\phi}{2}\right)^{(2\alpha+5)/2} \left(\cos\frac{\phi}{2}\right)^{(2\beta+3)/2}}\, d\phi$$

$$= O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) \int_{\delta}^{\pi-(1/n)} |F(\cos\phi) - A| \left(\cos\frac{\phi}{2}\right)^{(2\beta-1)/2} \cos\frac{\phi}{2}\, d\phi$$

$$+ O(n^{(2\alpha-1)/2}) \int_{\delta}^{\pi-(1/n)} |F(\cos\phi) - A| \left(\cos\frac{\phi}{2}\right)^{(2\beta-1)/2} d\phi$$

$$= O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) + O(n^{(2\alpha-1)/2}), \text{ by (3.5)}$$

$$= o(1) \text{ as } n \to \infty. \qquad \ldots(4.6)$$

We finally consider I_4.

$$I_4 = \int_{\pi-(1/n)}^{\pi} F(\phi)\, \frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{n} p_k \lambda_{n-k} P_{n-k}^{(\alpha+1,\beta)}(\cos\phi)\, d\phi$$

$$= \frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{n} \int_0^{1/n} |F(\pi-\phi)|\, \lambda_{n-k}\, p_k\, P_{n-k}^{(\alpha+1,\beta)}(\cos\phi)\, d\phi$$

(*equation continued on p.* 509)

$$= O\left(\frac{1}{P_n}\right) \sum_{k=0}^{n} p_k(n-k)^{\alpha+1} \int_0^{1/n} O(n-k)^{\beta} \mid F(-\cos\phi) - A \mid$$

$$\times \left(\sin\frac{\phi}{2}\right)^{2\beta+1} \left(\cos\frac{\phi}{2}\right)^{2\alpha+1} d\phi$$

$$= O\left(n^{\alpha+\beta+1}\right) \int_0^{1/n} \mid F(-\cos\phi) - A \mid \phi^{2\beta+1}\, d\phi$$

$$= O\left(n^{(2\alpha-1)/2}\right) \int_0^{1/n} \mid F(-\cos\phi) - A \mid \phi^{(2\beta-1)/2}\, d\phi$$

$$= o(1), \text{ by } (3.6)$$

Thus the theorem is completely established.

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