THE FLOW OF A CONDUCTING VISCOUS INCOMPRESSIBLE FLUID BETWEEN TWO PARALLEL PLATES UNDER A UNIFORM TRANSVERSE MAGNETIC FIELD

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In this paper the flow of a conducting viscous incompressible fluid between two parallel plates under a uniform transverse magnetic field is investigated under a pressure gradient (i) varying linearly with time and (ii) decreasing exponentially with time. As the intensity of the magnetic field increases, the velocity of the fluid is found to decrease in the first case. In this case it is also observed that the velocity at any point above the axis of channel is more than the velocity at its image point in the axis of the channel.

1. INTRODUCTION

In this paper we investigate the unsteady flow of a conducting viscous incompressible fluid between two parallel flat plates in the presence of a uniform transverse magnetic field. The upper plate is in uniform motion while the lower plate is at rest. In Part A of the paper the flow under pressure gradient varying linearly with time is discussed. In Part B the velocity distribution under the exponentially decreasing pressure gradient is studied. In the limiting case as the intensity of the magnetic field tends to zero, our results agree with those of Dube (1970). Taking the fluid to be of small conductivity with magnetic Reynolds number much less than unity, we neglect the induced magnetic field in comparison with the applied field (Sparow and Cess 1962, Verma and Mathur 1969, Bathaiah *et al.* 1975).

2. FORMULATION AND SOLUTION OF THE PROBLEM

The equations governing the motion of a conducting viscous incompressible fluid when the external forces and the induced magnetic field are neglected are

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q} \cdot \nabla)\, \bar{q} = - \frac{1}{\rho} \nabla P + \nu \nabla^2 \bar{q} + \frac{\mu_e}{\rho} (\bar{J} \times \bar{H}) \qquad \ldots(2.1)$$

$$\text{div}\, \bar{q} = 0 \qquad \ldots(2.2)$$

where $\bar{q}$ is the fluid velocity

$$\bar{J} = \sigma \mu_e (\bar{q} \times \bar{H})$$

P is the fluid pressure, ν the kinematic coefficient of viscosity, μ_e the magnetic permeability, $\bar{J}$ the current density, σ the electrical conductivity and $\bar{H}$ the magnetic field.

The axis of the channel is taken as the x-axis and a straight line perpendicular to that is taken as y-axis. The motion is two dimensional so that the velocity components u, v and w parallel to the coordinate axes are

$$u = u(x, y, t),\ v = 0,\ w = 0 \qquad \text{...(2.3}a\text{)}$$

$$P = P(x, y, t),\ \frac{\partial}{\partial z}(\ \ ) = 0. \qquad \text{...(2.3}b\text{)}$$

From (2.2) it follows that

$$\frac{\partial u}{\partial x} = 0 \text{ so that } u = u(y, t). \qquad \text{...(2.4)}$$

Using (2.3) and (2.4) in (2.1) we obtain

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - Ku \qquad \text{...(2.5)}$$

where

$$K = \frac{\sigma\mu_e^2\ H_0^2}{\rho}.$$

Also $\frac{\partial P}{\partial y} = 0$ so that $P = P(x, t)$. ...(2.6)

From eqns. (2.5) and (2.6) it follows that $\frac{\partial P}{\partial x}$ must be a constant or a function of time only, since P is not a function of y and u is not a function of x.

PART A

Variation of Pressure Gradient Linearly with Time

We assume that

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = a_0 + at. \qquad \text{...(2.7)}$$

Equation (2.5) then becomes

$$\frac{\partial u}{\partial t} = a_0 + at + \nu\frac{\partial^2 u}{\partial y^2} - Ku. \qquad \text{...(2.8)}$$

We define the Laplace transform

$$\bar{u} = \int_0^\infty e^{-st}\, u\, dt. \qquad \text{...(2.9)}$$

Equation (2.8) is transformed into

$$\frac{\partial^2 \bar{u}}{\partial y^2} - p^2\, \bar{u} = -\frac{1}{\nu}\left(u_0 + \frac{a_0}{s} + \frac{a}{s^2}\right) \qquad \text{...(2.10)}$$

where

$$p^2 = \frac{s + K}{\nu}$$

and u_0 is the initial value of u. Initially the pressure gradient is a_0 and the motion in the channel is steady.

Therefore

$$\frac{\partial u_0}{\partial t} = 0$$

and

$$\frac{d^2 u_0}{dy^2} - \frac{K u_0}{\nu} = -\frac{a_0}{\nu}. \qquad \text{...(2.11)}$$

The boundary conditions are

$$u_0 = 0 \text{ when } y = -y_0 \qquad \text{...(2.12}a\text{)}$$

and

$$u_0 = U \text{ when } y = y_0. \qquad \text{...(2.12}b\text{)}$$

The solution of (2.11) satisfying the boundary conditions (2.12) is

$$u_0 = \frac{U}{2}(A + B) + \frac{a_0}{K}(1 - B) \qquad \text{...(2.13)}$$

where

$$A = \frac{\sin h\,(\sqrt{K/\nu}\; y)}{\sin h\,(\sqrt{K/\nu}\; y_0)} \text{ and } B = \frac{\cos h\,(\sqrt{K/\nu}\; y)}{\cos h\,(\sqrt{K/\nu}\; y_0)}.$$

Using (2.13) in (2.10) we obtain

$$\frac{\partial^2 \bar{u}}{\partial y^2} - p^2 \bar{u} = -\frac{1}{\nu}\left[\frac{U}{2}(A + B) + \frac{a_0}{K}(1 - B) + \frac{a_0}{s} + \frac{a}{s^2}\right]. \qquad \text{...(2.14)}$$

The boundary conditions (2.12) are transformed into

$$\bar{u} = 0 \text{ when } y = -y_0 \qquad \text{...(2.15a)}$$

and

$$\bar{u} = \frac{U}{s} \text{ when } y = y_0. \qquad \text{...(2.15b)}$$

The solution of (2.14) satisfying (2.15) is

$$\bar{u} = \frac{U}{2s}(A + B) + \frac{a_0}{sK}(1 - B) + \frac{a}{s^2(s + K)} \times \left[1 - \frac{\cos h\,(\sqrt{s + K/\nu}\,y)}{\cos h\,(\sqrt{s + K/\nu}\,y_0)}\right]. \qquad \text{...(2.16)}$$

Inverting (2.16) we obtain

$$u = \frac{U}{2}(A + B) + \frac{a_0}{K}(1 - B) + \frac{a}{K^2}[(Kt - 1)(1 - B)]$$
$$- \frac{a}{2}\sqrt{\frac{K}{\nu}}\left[\frac{y \cos h\,(\sqrt{K/\nu}\,y_0) \sin h\,(\sqrt{K/\nu}\,y) - y_0 \cos h\,(\sqrt{K/\nu}\,y) \sin h\,(\sqrt{K/\nu}\,y_0)}{K^2 \cos h^2\,(\sqrt{K/\nu}\,y_0)}\right]$$
$$+ \frac{4a}{\pi} e^{-Kt} \sum_{n=0}^{\infty}\left[\frac{(-1)^n \exp\left\{-\frac{\nu\pi^2}{4y_0^2}(2n+1)^2 t\right\} \cos\left\{\frac{(2n+1)\pi y}{2y_0}\right\}}{(2n+1)\left\{K + \frac{\nu\pi^2}{4y_0^2}(2n+1)^2\right\}^2}\right]. \qquad \text{...(2.17)}$$

Since $u = u_0$ when $t = 0$ it follows from (2.13) and (2.17) that

$$\frac{4a}{\pi}\sum_{n=0}^{\infty}\left[\frac{(-1)^n \cos\left\{\frac{(2n+1)\pi y}{2y_0}\right\}}{(2n+1)\left\{K + \frac{\nu\pi^2}{4y_0^2}(2n+1)^2\right\}^2}\right] = \frac{a}{K^2}(1 - B)$$
$$+ \frac{a}{2}\sqrt{\frac{K}{\nu}}\left[\frac{y \cos h\,(\sqrt{K/\nu}\,y_0) \sin h\,(\sqrt{K/\nu}\,y) - y_0 \cos h\,(\sqrt{K/\nu}\,y) \sin h\,(\sqrt{K/\nu}\,y_0)}{K^2 \cos h^2\,(\sqrt{K/\nu}\,y_0)}\right]. \qquad \text{...(2.18)}$$

When $y = 0$, (2.18) gives

$$\frac{4a}{\pi}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(2n+1)\left\{K+\frac{\nu\pi^2}{4y_0^2}(2n+1)^2\right\}^2}\right]$$

$$= \frac{a}{K^2}\left[1 - \frac{1}{\cos h\,(\sqrt{K/\nu}\,y_0)}\right] - \frac{a}{2}\sqrt{\frac{K}{\nu}}\left\{\frac{y_0 \sin h\,(\sqrt{K/\nu}\,y_0)}{K^2 \cos h^2\,(\sqrt{K/\nu}\,y_0)}\right\}. \quad \ldots(2.19)$$

Introducing the non-dimensional quantities

$$W = \frac{2u}{U},\ T = \frac{Ut}{y_0},\ r = \sqrt{K/\nu}\,y,\ r_0 = \sqrt{K/\nu}\,y_0,\ b_0 = \frac{2a_0}{KU},$$

$$b = \frac{2a}{K^2U},\ c = \frac{\nu}{Ky_0^2},\ R = \frac{Uy_0}{\nu}$$

eqn. (2.17) can be expressed in the form

$$W = W_1 + W_2 \quad \ldots(2.20)$$

where

$$W_1 = \frac{\sinh r}{\sinh r_0} + \frac{\cosh r}{\cosh r_0} + \left\{1 - \frac{\cosh r}{\cosh r_0}\right\}(b_0 - b)$$
$$+ \frac{bT}{cR}\left(1 - \frac{\cosh r}{\cosh r_0}\right)$$
$$- \frac{b}{2}\left[\frac{r\cosh r_0 \sinh r - r_0 \cosh r \sinh r_0}{\cosh^2 r_0}\right]. \quad \ldots(2.21)$$

and

$$W_2 = \frac{4b}{\pi}\,e^{-T/Rc}\left[\frac{(-1)^n \exp\{-\pi^2(2n+1)^2\,T/4R\}\cos\{(2n+1)\,\pi r/2r_0\}}{(2n+1)\,\{1+[\pi^2 c\,(2n+1)^2]/4\}}\right]. \quad \ldots(2.22)$$

A graph of W_1 against r_0 is drawn (Fig. 1). It is found that W_1 decreases as r_0 increases. Since r_0 is proportional to $\sqrt{K}$, it follows that W_1 decreases as the intensity of magnetic field increases. Fixing r_0, r, b_0, b and c we have drawn a graph of W_1 against T (Fig. 2). It is observed that W_1 decreases as T increases. When $T > 1$, W_2 is very small when compared to W_1. Hence the transient part of fluid velocity is insignificant and W varies linearly with T when $T > 1$. It is also found that for fluids with small Reynolds number the transient part of W becomes insignificant faster than for fluids with large Reynolds number.

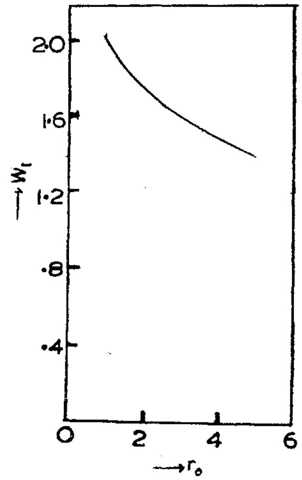

FIG. 1.

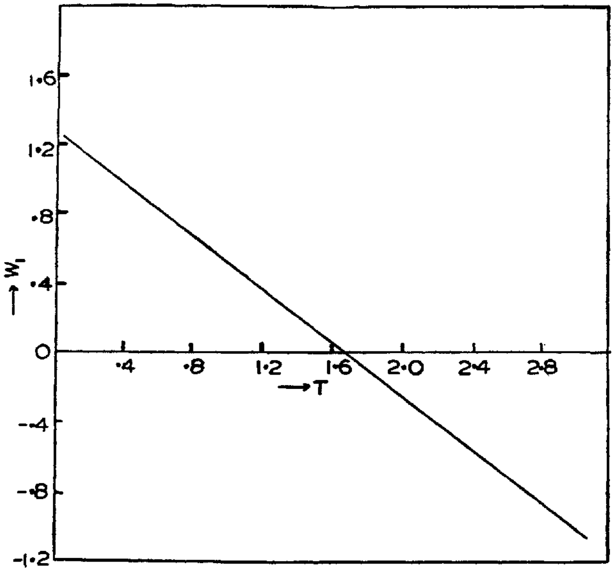

FIG. 2.

PART B

EXPONENTIALLY DECREASING PRESSURE GRADIENT

Taking

$$-\frac{1}{\rho}\frac{\partial P}{\partial x} = a_0 + \sum_{m=1}^{\infty} a_m e^{-mt}. \qquad ...(2.23)$$

We reduce eqn. (2.5) tc

$$\frac{\partial u}{\partial t} = a_0 + \sum_{m=1}^{\infty} a_m e^{-mt} + \nu\frac{\partial^2 u}{\partial y^2} - Ku. \qquad ...(2.24)$$

The Laplace transform of eqn. (2.24) is

$$\frac{\partial^2 \bar{u}}{\partial y^2} - p^2\bar{u} = -\frac{1}{\nu}\left[u_0 + \frac{a_0}{s} + \sum_{m=1}^{\infty}\frac{a_m}{s+m}\right] \qquad ...(2.25)$$

where u_0 is again given by (2.13).

The solution of eqn. (2.25) satisfying the boundary conditions (2.15) is

$$\bar{u} = \frac{U}{2s}(A + B) - \frac{a_0 B}{Ks} + \frac{a_0}{Ks} + \sum_{m=1}^{\infty} \frac{a_m}{(s + K)(s + m)} \left\{ 1 - \frac{\cosh\left(\sqrt{(s + K/\nu)}\,y\right)}{\cosh\left(\sqrt{(s + K/\nu)}\,y_0\right)} \right\}. \quad \ldots(2.26)$$

Inverting we obtain

$$u = W'_1 + W'_2 + W'_3 + W'_4 \quad \ldots(2.27)$$

where

$$W'_1 = \frac{U}{2}(A + B) \quad \ldots(2.28a)$$

$$W'_2 = \frac{a_0}{K}(1 - B) \quad \ldots(2.28b)$$

$$W'_3 = -\sum_{m=1}^{\infty} \frac{a_m e^{-mt}}{m - K} \left[1 - \frac{\cosh\{(\sqrt{(K - m)/\nu})\,y\}}{\cosh\{(\sqrt{(K - m)/\nu})\,y_0\}} \right] \quad \ldots(2.28c)$$

$$W'_4 = \frac{4}{\pi} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left[\frac{(-1)^n a_m \exp[-\{K + [\nu\pi^2(2n + 1)^2]/4y_0^2\}\,t] \cos\{[(2n + 1)\pi y]/2y_0\}}{(2n + 1)\left\{ m - K - \frac{\nu\pi^2}{4y_0^2}(2n + 1)^2 \right\}} \right]. \quad \ldots(2.28d)$$

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