STRESS ANALYSIS IN FINITE CURVED BEAMS UNDER GENERAL EQUILIBRIUM LOADING

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The paper presents an elastostatic boundary value problem associated with finite curved beams undergoing generalized plane stress. For a beam under general equilibrium loading, it is shown how suitable biharmonic functions can be superposed to make each of the boundary loading self-equilibrating. The loads on the curved edges are transferred to the radial edges by developing a Fourier Series solution. The radial eigenfunctions developed in an earlier paper (Sarma *et al.* 1975) are used to provide solutions for the residual problem. Numerical solutions are presented for illustrating the solutions for some boundary loading.

1. INTRODUCTION

The problem of determining the stress and displacement distributions in a rectangle under general equilibrium loading has been the subject of several investigators in the literature. Gaydon (1968) has discussed this problem in full generality and has presented solutions for some boundary loadings. The participation constants involved in the eigenfunction expansions have been determined by expanding each of the non-orthogonal eigenfunctions in terms of an orthogonal set of clamped beam functions (Gaydon and Shepherd 1964).

The present paper is an attempt to extend Gaydon's analysis for finite curved beams in general equilibrium under continuous boundary loadings. In the first part of the paper analysis is presented for the determination of a stress function Φ, the stresses derived from which when subtracted from the original distribution will make the residual stresses on each of the boundaries self equilibrating. In the second part of Fourier series solution ψ in the angular co-ordinate is developed for the biharmonic equation, the stresses derived from which when subtracted from the existing self-equilibrating stresses will make the curved boundaries free of tractions. Finally the analysis presented by Sarma *et al.* (1975) has been used to determine a stress function χ to account for the self equilibrating loads existing on the radial edges. The superposition of the three stress function viz., $\chi + \psi + \Phi$ will constitute the stress function for the prescribed boundary loadings.

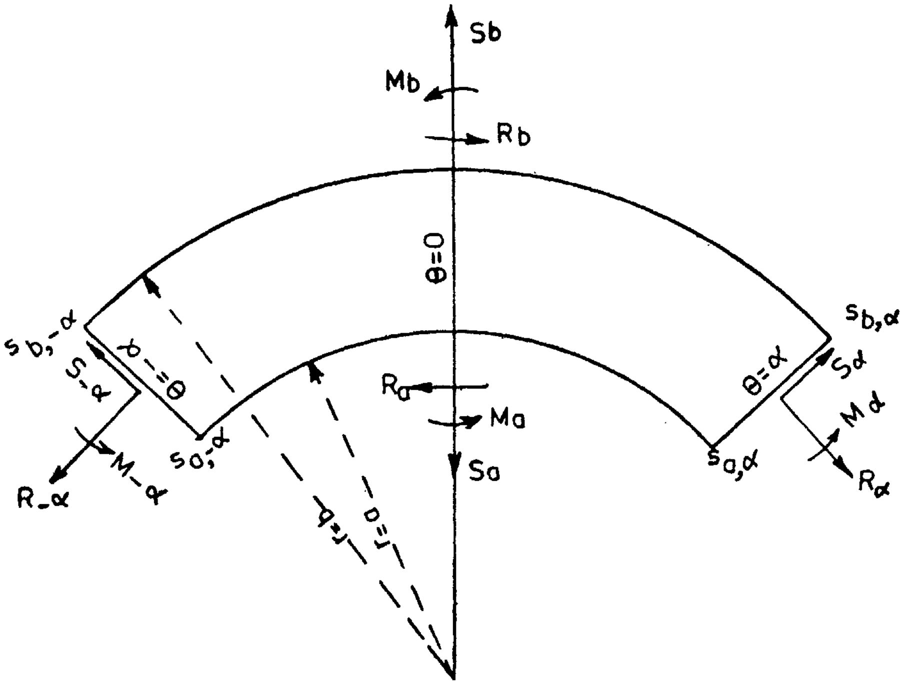

FIG. 1. Finite curved beam under general equilibrium loading.

2. STRESS FUNCTION Φ FOR GIVEN RESULTANT LOADS

Consider a curved beam (Fig. 1) occupying the region $a \leqslant r \leqslant b$, $-\alpha \leqslant \theta \leqslant \alpha$ and undergoing generalised plane stress. We assume that it is in equilibrium under the action of distributed loads along its boundary. We denote the normal stresses by σ_r, σ_θ and the shear stress by τ. The subscripts $a, b, \alpha, -\alpha$ appearing on these stress components will indicate the boundary along which the stresses are acting. Further let $s_{a,\alpha}$, $s_{a,-\alpha}$, $s_{b,-\alpha}$ and $s_{b,\alpha}$ respectively denote the shear stresses at the corners (a, α), $(a, -\alpha)$, $(b, -\alpha)$ and $(b, +\alpha)$.

The resultant loads on the boundary $r = a$ are given by

$$R_a = \int_{-\alpha}^{\alpha} a(\sigma_{ra} \sin\theta + \tau_a \cos\theta)\, d\theta \qquad \text{...(2.1}a\text{)}$$

$$S_a = \int_{-\alpha}^{\alpha} a(\sigma_{ra} \cos\theta - \tau_a \sin\theta)\, d\theta \qquad \text{...(2.1}b\text{)}$$

$$M_a = a^2 \int_{-\alpha}^{\alpha} \tau_a\, d\theta. \qquad \text{...(2.1}c\text{)}$$

The resultant loads on $r = b$ are given by

$$R_b = \int_{-\alpha}^{\alpha} b(\sigma_{rb} \sin\theta + \tau_b \cos\theta)\, d\theta \qquad \ldots(2.2a)$$

$$S_b = \int_{-\alpha}^{\alpha} b(\sigma_{rb} \cos\theta + \tau_b \sin\theta)\, d\theta \qquad \ldots(2.2b)$$

$$M_b = -\, b^2 \int_{-\alpha}^{\alpha} \tau_b\, d\theta. \qquad \ldots(2.2c)$$

The resultant loads on $\theta = \alpha$ are given by

$$R_\alpha = \int_{a}^{b} \sigma_{\theta\alpha}\, dr \qquad \ldots(2.3a)$$

$$S_\alpha = \int_{a}^{b} \tau_\alpha\, dr \qquad \ldots(2.3b)$$

$$M_\alpha = -\int_{a}^{b} r\sigma_{\theta\alpha}\, dr. \qquad \ldots(2.3c)$$

Finally the resultant loads on $\theta = -\,\alpha$ are given by

$$R_{-\alpha} = \int_{a}^{b} \sigma_{\theta(-\alpha)}\, dr \qquad \ldots(2.4a)$$

$$S_{-\alpha} = \int_{a}^{b} \tau_{(-\alpha)}\, dr \qquad \ldots(2.4b)$$

$$M_{-\alpha} = \int_{a}^{b} r\sigma_{\theta(-\alpha)}\, dr. \qquad \ldots(2.4c)$$

It is easy to verify that the equations of equilibrium in terms of the stress resultants and moments can be written as follows:

$$(R_\alpha - R_{-\alpha})\cos\alpha + (S_\alpha + S_{-\alpha})\sin\alpha + R_b - R_a = 0 \qquad \ldots(2.5a)$$

$$(S_\alpha - S_{-\alpha})\cos\alpha - (R_\alpha + R_{-\alpha})\sin\alpha + S_b - S_a = 0 \qquad \ldots(2.5b)$$

$$M_a + M_b + M_\alpha + M_{-\alpha} = 0. \qquad \ldots(2.5c)$$

For determining a stress function Φ which has the same resultant loads as the prescribed distribution and the same shear stress at each of the corners, we split Φ into two parts depending upon whether it is even or odd in the angular co-ordinate. Throughout the paper the superscript '*e*' or '*o*' will be used to denote that the function under consideration is even or odd with respect to θ.

Stress Function Φ^e even in θ

In this case σ_r and σ_θ are even in θ and τ is odd in θ. Hence it follows that

$$\left.\begin{aligned} & R^e_a = R^e_b = 0,\ M^e_a + M^e_b = 0,\ R^e_{-\alpha} = R^e_\alpha \\ & S^e_\alpha = -\ S^e_{-\alpha},\ s^e_{a,\alpha} = -\ s^e_{b,-\alpha},\ s^e_{b,\alpha} = -\ s^e_{a,-\alpha}. \end{aligned}\right\} \quad \text{...(2.6}a\text{)}$$

The equilibrium equations can be rewritten as

$$M^e_\alpha + M^e_{-\alpha} = 0 \quad \text{...(2.6}b\text{)}$$

and

$$2S^e_\alpha \cos\alpha - 2R^e_\alpha \sin\alpha + S^e_b - S^e_a = 0. \quad \text{...(2.6}c\text{)}$$

We take

$$\Phi^e = (c_1 + c_2 r^2 + c_3/r^2)\cos 2\theta + c_4 r^2 + c_5 \ln r + c_6 r^2 \ln r. \quad \text{...(2.7)}$$

The constants c_i $(i = 1, 6)$ are determined using six independent conditions given by (2.6). Indeed six linear algebraic equations in the unknowns c_i $(i = 1, 6)$ can be expressed in terms of six independent quantities—say $s^e_{a,\alpha}$, $s^e_{b,\alpha}$, s^e_b, S^e_a, S^e_α and M^e_α. The explicit determination of these constants though straight-forward involves heavy algebraic work. The expressions determining these constants being very unwieldy they are not presented here.

Stress Function Φ^o odd in θ

In this case σ_r and σ_θ are odd in θ and τ is even in θ. Hence it follows that

$$\left.\begin{aligned} & S^o_b = S^o_a = 0,\ R^o_\alpha = -\ R^o_{-\alpha},\ S^o_\alpha = S^o_{-\alpha}, \\ & M^o_{-\alpha} = M^o_\alpha,\ s^o_{a,\alpha} = s^o_{b,-\alpha},\ s^o_{b,\alpha} = s^o_{a,-\alpha}. \end{aligned}\right\} \quad \text{...(2.8}a\text{)}$$

The equilibrium equations can be rewritten as

$$M^o_a + M^o_b = 0 \quad \text{...(2.8}b\text{)}$$

$$2R^o_\alpha \cos\alpha + 2S^o_\alpha \sin\alpha - R^o_b - R^o_a = 0. \quad \text{...(2.8}c\text{)}$$

We take

$$\Phi^0 = (c_7 + c_8 r^2 + c_9/r^2)\sin 2\theta + (c_{10}\ln r + c_{11} r^2 \ln r + c_{12} r^2)\,\theta + c_{13} r\theta\cos\theta \quad \text{...(2.9)}$$

The constant c_i $(i = 7, 13)$ are determined using seven independent quantities appearing in the relations (2.8). For example they can be expressed in terms of R^o_a, R^o_α, $s^o_{a,\alpha}$, $s^o_{b,\alpha}$, S^o_α, M^o_α and M^o_a. For reasons mentioned above, these constants are not presented here.

Given a general distributed loading over all the four edges of the beam which have the stress resultants and moments with shear stresses at each corner as shown in Fig. 1, the constants c_i $(i = 1, 13)$ are determined and the stresses derived from the stress function Φ are subtracted from the given stress distribution on the boundaries. Then the remaining distributed normal and shear stresses on each side will be self-equilibrating. Further, the shear stress at each corner will be zero.

The problem now reduces to finding the stresses in a finite curved beam with self-equilibrating stresses on each of the sides as follows :

$$\sigma_r(\theta) = \sigma_{rb}(\theta);\ \tau(\theta) = \tau_b(\theta) \text{ on } r = b \qquad \text{...(2.10}a)$$

$$\sigma_r(\theta) = \sigma_{ra}(\theta);\ \tau(\theta) = \tau_a(\theta) \text{ on } r = a \qquad \text{...(2.10}b)$$

$$\sigma_\theta(r) = \sigma_{\theta\alpha}(r);\ \tau(r) = \tau_\alpha(r) \text{ on } \theta = \alpha \qquad \text{...(2.10}c)$$

$$\sigma_\theta(r) = \sigma_{\theta(-\alpha)}(r);\ \tau(r) = \tau_{(-\alpha)}(r) \text{ on } \theta = -\alpha. \qquad \text{...(2.10}d)$$

It should be noted that the stress function Φ^e and Φ^o given in (2.7) and (2.9) are not unique. Instead any stress function can be conveniently used and different stress functions will yield a different residual problem to solve.

3. Stress Function ψ as a Fourier Series in the Angular Co-ordinate

We assume the solution of the biharmonic equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0 \qquad \text{...(3.1)}$$

in the form

$$\psi = \sum_{m=1}^{\infty} f_m(r) \frac{\cos}{\sin} (\alpha_m \theta) \qquad \text{...(3.2)}$$

where

$$\alpha_m = \alpha^e_m = \frac{m\pi}{2\alpha}$$

or

$$\alpha_m = \alpha^o_m = (2m - 1)\frac{\pi}{2\alpha} \qquad \text{...(3.3)}$$

depending upon whether the stress distribution on the curved edges $r = a$ and $r = b$ is even or odd in θ.

Substituting (3.3) and (3.2) in (3.1) we get a fourth order ordinary differential equation for the function $f_m(r)$, the solution for which in the general form may be taken as

$$f_m(r) = A_m\, r^{\alpha_m} + B_m\, r^{-\alpha_m} + C_m\, r^{\alpha_m+2} + D_m\, r^{-\alpha_m+2}.$$

If the four boundary tractions σ_{ra}, σ_{rb}, τ_a and τ_b are now expanded in a Fourier series (cosine series or sine series depending upon whether the functions are even or odd) and are equated to the stresses determined using the stress function (11), we get four linear algebraic equations for the unknowns A_m, B_m, C_m and D_m. The analysis being straightforward, the details are not presented here. If the stresses derived from the function are subtracted from the existing self-eqilibrating stresses, the residual load will leave the curved boundaries 'stress free' with zero shear stresses at each of the corners and self equilibrating stresses on the radial edges. The analysis for this residual problem has been carried out using the method of biorthogonal functions developed by Johnson and Little (1964). In a recent paper (Sarma *et al.* 1975) this method has been used to study the end effects in a long curved beam. The analysis for a finite curved beam verbatim follows on the same lines as in Sarma *et al.* (1975). We have, therefore, preferred to present only the solutions of the title problem for some boundary loadings.

4. Numerical Examples

Example 1 : *Curved beam subjected to uniform pressure on the curved edges*

Consider the boundary loading

$$\sigma_{ra} = -\frac{b}{a}\, p_0, \quad \sigma_{rb} = -p_0$$

$$\tau_a = \tau_b = \sigma_{\theta\alpha} = \sigma_{\theta(-\alpha)} = \tau_\alpha = \tau_{(-\alpha)} = 0.$$

The stress function (2.7) can be suggested for this case. Using the analysis presented earlier the constants c_i $(i = 1, 6)$ are calculated. The use of the stress function (2.7) leaves self equilibrating tractions σ_θ on the radial edges $\theta = \pm\, \alpha$, the rest of the stress components on all the boundaries being zero. In this case since the curved boundaries are stress free, we can proceed to determine the stress function χ which will give rise to the self equilibrating load on the radial edges. In practice the end effects on account of this load are neglected by invoking the Saint Venant's principle. The end effects have been studied and a comparative study of the two results has been made for $b/a = 3.0$ and $\alpha = \pi/3$. These are graphically illustrated in Figs. 2 and 3.

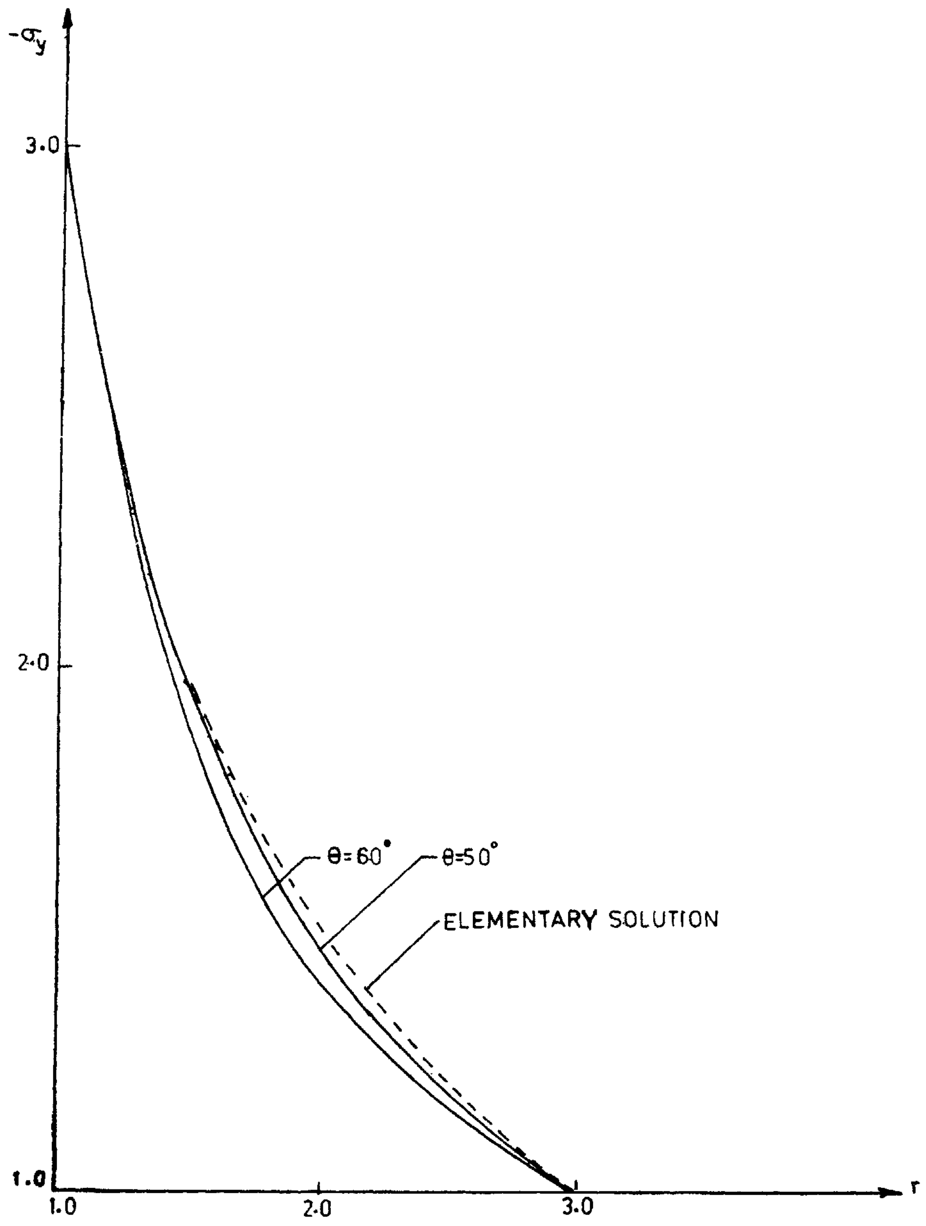

FIG. 2. Curved beam under uniform pressure. Comparison of the normal stress σ_r with the elementary solution.

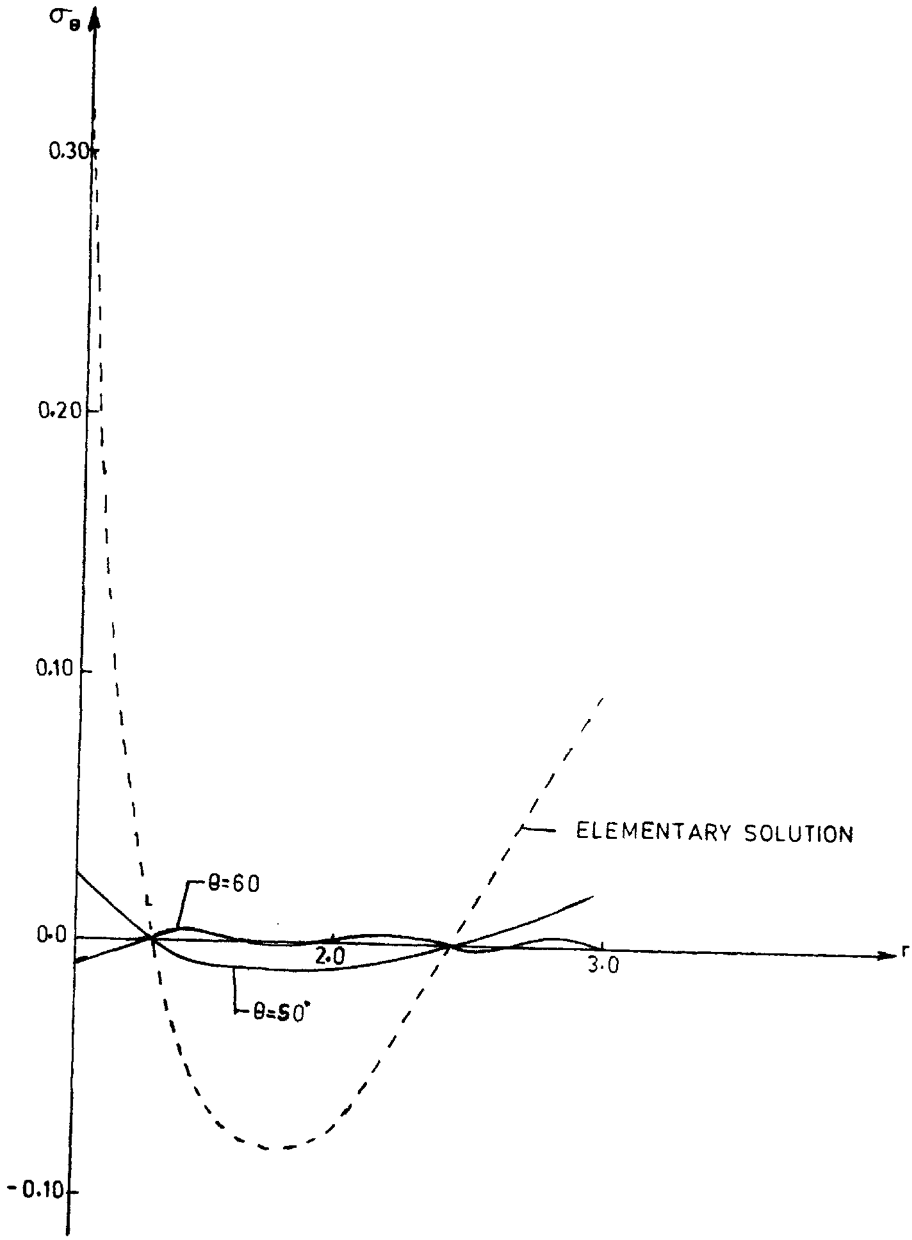

FIG. 3. Curved beam under uniform pressure. Comparison of the normal stress σ_θ with the elementary solution.

Example 2

Consider the boundary loading

$$\sigma_{ra} = b \cos 2\theta, \quad \sigma_{rb} = a \cos 2\theta$$

$$\tau_a = \tau_b = 0$$

$$\sigma_{\theta\alpha} = \sigma_{\theta(-\alpha)} = kr^2$$

$$\tau_\alpha = (r - a)(b - r), \quad \tau_{(-\alpha)} = -(r - a)(b - r).$$

The constant k is adjusted such that the beam is in equilibrium. It is given by

$$k = \frac{(b - a)^2 \cot \alpha}{2(a^2 + ab + b^2)}.$$

The non-vanishing stress resultants and moments are given by

$$S^e_\alpha = \tfrac{1}{6}(b - a)^3, \quad R^e_\alpha = \tfrac{1}{3} kr^3, \quad M^e_\alpha = \tfrac{1}{4} kr^4$$

$$S^e_a = S^e_b = ab\,(\sin \alpha + \tfrac{1}{3} \sin 3\alpha).$$

Using these expressions, the constants c_i $(i = 1, 6)$ are determined. The residual stresses are self equilibrating. The stress function ψ has been determined using 20 terms of the Fourier expansions. The eigenfunction analysis for the self equilibrating stresses on the radial edges has been carried out using the first 5 pairs of eigenvalues. From the stress function $\psi + \Phi + \chi$, the stresses on all the four boundaries are calculated and the accuracy in reproducing the prescribed boundary stresses are presented in Tables I and II for aspect ratio $b/a = 2.0$ and $\alpha = \pi/3$.

TABLE I

Reproduction of the normal stresses on the edges $r = a$ and $r = b$

	Normal stress σ_r			
	specified function σ_r on $r = a$	calculated σ_r on $r = a$	specified function σ_r on $r = b$	calculated σ_r on $r = b$
0°	2.0000	2.0639	1.0000	1.0149
10°	1.8793	1.9535	0.9396	0.9387
20°	1.5320	1.4969	0.7660	0.7657
30°	1.0000	0.8935	0.5000	0.4870
40°	0.3472	0.3044	0.1736	0.1783
50°	− 0.3472	− 0.2024	− 0.1736	− 0.1641
60°	− 1.0000	− 0.8920	− 0.5000	− 0.5196

TABLE II

Reproduction of the normal and shear stresses on the edge $\theta = \alpha$

r	Normal stress σ_θ specified function σ_θ on $\theta = \alpha$	Normal stress σ_θ calculated σ_θ on $\theta = \alpha$	Shear stress τ specified function τ on $\theta = \alpha$	Shear stress τ calculated τ on $\theta = \alpha$
1.0	0.0412	0.0545	0.0000	0.0000
1.1	0.0498	0.0584	0.0900	0.0989
1.2	0.0593	0.0526	0.1600	0.1555
1.3	0.0696	0.0736	0.2100	0.2111
1.4	0.0808	0.0779	0.2400	0.2408
1.5	0.0927	0.0958	0.2500	0.2460
1.6	0.1055	0.1046	0.2400	0.2467
1.7	0.1191	0.1132	0.2100	0.2075
1.8	0.1336	0.1392	0.1600	0.1521
1.9	0.1488	0.1488	0.0900	0.0953
2.0	0.1649	0.1728	0.0000	0.0000

5. CONCLUDING REMARKS

The paper makes it possible to solve any finite curved beam problem in generalized plane stress for which the boundary conditions are stress conditions. It is assumed that there is no discontinuity of the stress on an edge and a discontinuity of the shear stress at a corner. This is assumed for the satisfactory application of the earlier paper by Sarma *et al.* (1975). The method described in this paper makes it possible to obtain the stresses near a boundary where the load is applied and hence obtain solutions of problems which do not come within the scope of Saint Venant's principle.

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