QUASI-STATIC THERMAL DEFLECTION OF A THIN CLAMPED CIRCULAR PLATE SUBJECTED TO RANDOM TEMPERATURE DISTRIBUTION ON ITS UPPER FACE

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The present paper deals with the problem of the quasi-static thermal deflection of a thin elastic circular plate fixed and clamped at its edge and subject to random heat inputs on its face.

1. INTRODUCTION

Previously some workers Boley and Weiner (1960), Nowacki (1962), Roy Choudhuri (1973) on thermoelasticity have studied the deflection of plates under various deterministic types of temperature distribution on its faces. But as yet nobody has studied the problems for random temperature distribution. In this paper, the thermal deflection of a thin clamped circular plate subjected to random temperature distribution on one face is discussed.

2. FUNDAMENTAL EQUATION AND SOLUTION OF HEAT CONDUCTION EQUATION

Let us consider a circular plate of thickness h occupying the space $0 \leqslant r \leqslant a$, $-\frac{h}{2} \leqslant z \leqslant +\frac{h}{2}$.

Heat conduction equation is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{K}\frac{\partial T}{\partial t} \qquad \ldots(1)$$

where K is the thermal diffusivity.

The initial and boundary conditions are

$$[T]_{z=-h/2} = 0, \qquad [T]_{t=0} = 0$$

$$\left[\frac{\partial T}{\partial r}\right]_{r=a} = 0, \qquad T = \psi(t)\,\phi(r)$$

on the circular region $0 < r < r_0$ of the upper face $z = h/2$, Taking Laplace transform of (1), we get

$$\frac{d^2\bar{T}}{dr^2} + \frac{1}{r}\frac{d\bar{T}}{dr} + \frac{d^2\bar{T}}{dz^2} = \frac{p}{k}\bar{T} \qquad \text{...(2)}$$

where p is the transform parameter

$$(\bar{T})_{z=-h/2} = 0, \quad \left(\frac{d\bar{T}}{dr}\right)_{r=a} = 0 \qquad \text{...(3)}$$

$$(\bar{T})_{z=h/2} = \bar{\psi}(p)\,\phi(r). \qquad \text{...(4)}$$

We assume

$$\bar{T}(r, z, p) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \sinh\left[r\gamma_n\left(z + \frac{h}{2}\right)\right] \qquad \text{...(5)}$$

as a solution of (2) satisfying (3), where $\alpha_1, \alpha_2, \ldots$ are the roots of

$$J_1(\alpha a) = 0.$$

Equation (2) becomes

$$r\gamma_n = [\alpha_n^2 + (p/k)]^{1/2}, \; n = 1, 2, 3, \ldots$$

Again we assume

$$\phi(r) = \sum_{n=1}^{\infty} B_n J_0(\alpha_n r).$$

$$\therefore \quad B_n \int_0^a r\,[J_0(\alpha_n r)]^2\,dr = \int_0^a f(r) r J_0(\alpha_n r)\,dr$$

$$= \int_0^{r_0} r J_0(\alpha_n r)\,dr.$$

Since

$$\phi(r) = 1 \text{ for } 0 < r < r_0$$
$$= 0 \text{ for } r > r_0$$

hence

$$B_n = \frac{2r_0\,J_1(\alpha_n r_0)}{a^2\,\alpha_n[J_0(\alpha_n a)]^2}. \qquad \text{...(6)}$$

From (4) and (5)

$$\bar{\psi}(p)\,\phi(r) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \sinh(r\gamma_n h).$$

Hence

$$A_n = \frac{\bar{\psi}(p)\,\phi(r)}{J_0(\alpha_n r)\sinh(r\gamma_n h)} \qquad \ldots(7)$$

with the help of (6) and (7), (5) becomes

$$\bar{T}(r, z, p) = \frac{2r_0}{a^2}\sum_{n=1}^{\infty} \frac{\bar{\psi}(p) J_0(\alpha_n r)\, J_1(\alpha_n r_0)\ \sinh\left\{\left(\alpha_n^2 + \frac{p}{k}\right)^{1/2}\left(z + \frac{h}{2}\right)\right\}}{\alpha_n\{J_0(\alpha_n a)\}^2 \sinh\left\{\left(\alpha_n^2 + \frac{p}{k}\right)^{1/2} h\right\}}. \qquad \ldots(8)$$

Using the inversion theorem of t defined by

$$F(t) = \frac{1}{2\pi i}\int_{x_0 - i\infty}^{x_0 + i\infty} e^{pt}\bar{F}(p)\,dp.$$

Let

$$I = \frac{1}{2\pi i}\int_{x_0 - i\infty}^{x_0 + i\infty} \frac{\sinh\left\{\left(\alpha_n^2 + \frac{p}{k}\right)^{1/2}\left(z + \frac{h}{2}\right)\right\}}{\sinh\left\{\left(\alpha_n^2 + \frac{p}{k}\right)^{1/2} h\right\}} \cdot e^{pt}\,dp.$$

$\sinh\left[\left(\alpha_n^2 + \frac{p}{k}\right)^{1/2} h\right] = 0$ gives

$$p = p_n = -\,k\left[\alpha_n^2 + \frac{m^2\pi^2}{h^2}\right]$$

where

$$m = 0, 1, 2, 3, \ldots .$$

Residue at p_m

$$= \lim_{p \to p_m}\left[\frac{(p - p_m)\,e^{pt}\ \sinh\left(\alpha_n^2 + \frac{p}{k}\right)^{1/2}\left(z + \frac{h}{2}\right)}{\sinh\left\{\left(\alpha_n^2 + \frac{p}{k}\right)^{1/2} h\right\}}\right.$$

$$= -\,(-\,1)^m\,\frac{2\pi m}{kh^2}\ \sin\left\{\frac{m\pi}{h}\left(z + \frac{h}{2}\right)\right\}$$

$$\times \exp\left[-\,k\{(m^2\pi^2/h^2) + \alpha_n^2\}t\right].$$

Hence the temperature distribution is

$$T(r, z, t) = \frac{-4\pi r_0}{kh^2a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^m \cdot \frac{mJ_0(\alpha_n r)\, J_1(\alpha_n r_0)}{\alpha_n\{J_0(\alpha_n a)\}^2} \sin\left\{\beta_m\left(z + \frac{h}{2}\right)\right\}$$
$$\times \int_0^t \psi(\tau)\, e^{-\gamma_{mn}(t-\tau)}\, dN(\tau). \qquad \ldots(9)$$

where $dN(\tau)$ represents the number of pulses in the interval $(\tau, \tau + d\tau)$,

$$\beta_m = \frac{m\pi}{h} \text{ and } \gamma_{mn} = k(\beta_m^2 + \alpha_n^2).$$

3. Moments and Correlation of Temperature Distribution

We assume, according to the Poisson's distribution, in case of product density of degree one,

$\epsilon\{dN(\tau)\} = Cd\tau$, where C is the constant average density

$\epsilon\{\psi(\tau)\} = D$, a constant.

Since the probability of occurrence of one pulse in $d\tau$ is proportional to $d\tau$ but for more than one pulse in $d\tau$ is of negligible smaller order of magnitude than $d\tau$. Hence the moment of temperature distribution is given by

$$\epsilon\{T(r, z, t)\} = \frac{-4\pi r_0}{kh^2a^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^m \frac{mCD\, J_0(\alpha_n r)\, J_1(\alpha_n r_0)}{\alpha_n\{J_0(\alpha_n a)\}^2}$$
$$\times \sin\left\{\beta_m\left(z + \frac{h}{2}\right)\right\} \frac{1 - \exp(-\gamma_{mn} t)}{\gamma_{mn}}. \qquad \ldots(10)$$

In case of product density of degree two, we assume

$\epsilon\{dN(\tau_1)\, dN(\tau_2)\} = C^2$, where C is the constant average density

$\epsilon\{\psi(\tau_1)\, \psi(\tau_2)\} = \exp(-\mu(\tau_2 - \tau_1)),\ \mu > 0,\ \tau_2 > \tau_1.$

Hence the correlation of temperature distribution is given by

$$\epsilon\{T(r, z, t_1)\, T(r, z, t_2)\} = \frac{16\pi^2 r_0^2}{k^2h^4a^4}$$
$$\times \sum_{m, m'} \sum_{n, n'} (-1)^{m+m'} \frac{mm'\, J_0(\alpha_n r)\, J_0(\alpha_{n'} r)\, J_1(\alpha_n r_0)\, J_1(\alpha_{n'} r_0)}{\alpha_n \alpha_{n'}\{J_0(\alpha_n a)\}^2\, \{J_0(\alpha_{n'} a)\}^2}$$
$$\times \sin\left\{\beta_m\left(z + \frac{h}{2}\right)\right\} \sin\left\{\beta_{m'}\left(z + \frac{h}{2}\right)\right\} \times$$

(*equation continued on p.* 545)

$$\times\Big[C^2 \int_0^{t_1} d\tau_1 \int_{\tau_1}^{t_2} \exp\{-\mu(\tau_2-\tau_1) - \gamma_{mn}(t_1-\tau_1) - \gamma_{m'n'}(t_2-\tau_2)\}\cdot d\tau_2$$

$$+ C^2 \int_0^{t_1} d\tau_2 \int_{\tau_2}^{t_1} \exp\{-\mu(\tau_1-\tau_2) - \gamma_{mn}(t_1-\tau_1) - \gamma_{m'n'}(t_2-\tau_2)\}\cdot d\tau_1$$

$$+ \int_0^{t_1} \exp\{-\gamma_{mn}(t_1-\tau_1) - \gamma_{m'n'}(t_2-\tau_1)\}\cdot d\tau_1\Big]$$

$$= \frac{16\pi^2 r_0^2}{k^2 h^4 a^4}$$

$$\times \sum_{m,\,m'} \sum_{n,\,n'} (-1)^{m+m'} \frac{mm'\, J_0(\alpha_n r)\, J_0(\alpha_{n'} r)\, J_1(\alpha_n r_0)\, J_1(\alpha_{n'} r_0)}{\alpha_n \alpha_{n'} \{J_0(\alpha_n a)\}^2 \{J_0(\alpha_{n'} a)\}^2}$$

$$\times \left\{ \sin\beta_m \left(z + \frac{h}{2}\right)\right\} \left\{ \sin\beta_{m'} \left(z + \frac{h}{2}\right)\right\}$$

$$\times\Big[\frac{C^2}{\gamma_{m'n'}-\mu}\Big\{\frac{\exp(-\mu(t_2-t_1)) - \exp(-\mu t_2 - \gamma_{mn} t_1)}{\mu+\gamma_{mn}}$$

$$- \frac{\exp(-\gamma_{m'n'}(t_2-t_1)) - \exp(-\gamma_{mn} t_1 - \gamma_{m'n'} t_2)}{\gamma_{mn}+\gamma_{m'n'}}\Big\}$$

$$+ \frac{C^2}{\gamma_{mn}-\mu}\Big\{\frac{\exp(-\gamma_{m'n'}(t_2-t_1)) - \exp(-\mu t_1 - \gamma_{m'n'} t_2)}{\mu+\gamma_{m'n'}}$$

$$- \frac{\exp(-\gamma_{m'n'}(t_2-t_1)) - \exp(-\gamma_{mn} t_1 - \gamma_{m'n'} t_2)}{\gamma_{mn}+\gamma_{m'n'}}\Big\}$$

$$+ \frac{1}{\gamma_{mn}+\gamma_{m'n'}}\Big\{\exp(-\gamma_{m'n'}(t_2-t_1)) - \exp(-\gamma_{mn} t_1 - \gamma_{m'n'} t_2)\Big\}\Big] \quad \ldots(11)$$

4. Quasi-static Thermal Deflection Subject to Random Pulses

$$M_r(r,t) = \alpha E \int_{-h/2}^{h/2} z T(r,z,t)\, dz$$

$$= \frac{4 r_0 \alpha E}{a^2 k} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)\, J_1(\alpha_n r_0)}{\alpha_n \{J_0(\alpha_n a)\}^2} \int_0^t \Psi(\tau) \exp(-\gamma_{mn}(t-\tau))\cdot dN(\tau).$$

The differential equation satisfied by the thermal deflection $w(r, t)$ is

$$D\nabla_1^4 w = -\frac{\nabla_1^2 M_T}{(1-\nu)}.$$

Since the edge of the circular plate is fixed and clamped,

$$w = \frac{\partial w}{\partial r} = 0 \text{ at } r = a$$

hence

$$w(r, t) = \frac{4r_0\alpha E}{a^2kD(1-\nu)} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\{J_0(\alpha_n r) - J_0(\alpha_n a)\}\, J_1(\alpha_n r_0)}{\alpha_n^3 \{J_0(\alpha_n a)\}^2}$$

$$\times \int_0^t \Psi(\tau) \exp\left(-\gamma_{mn}(t-\tau)\right) dN(\tau). \qquad \ldots(12)$$

Moment of thermal deflection is given by

$$\epsilon\{w(r, t)\} = \frac{4r_0\alpha E}{a^2kD(1-\nu)} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{C'D'\{J_0(\alpha_n r) - J_0(\alpha_n a)\}\, J_1(\alpha_n r_0)}{\alpha_n^3 \{J_0(\alpha_n a\}^2}$$

$$\times \frac{1 - e^{-\gamma_{mn}t}}{\gamma_{mn}}. \qquad \ldots(13)$$

Correlation of thermal deflection is given by

$$\epsilon\{w(r, t_1)\, w(r, t_2)\} = \frac{16r_0^2\,\alpha^2E^2}{a^4k^2D^2(1-\nu)^2}$$

$$\times \sum_{m,m'}\sum_{n,n'} \frac{\{J_0(\alpha_n r) - J_0(\alpha_n a)\}\,\{J_0(\alpha_{n'} r) - J_0(\alpha_{n'} a)\}\, J_1(\alpha_n r_0)\, J_0(\alpha_{n'} r_0)}{\alpha_n^3\ \alpha_{n'}^3\, \{J_0(\alpha_n a)\}^2\, \{J_0(\alpha_{n'} a)\}^2}$$

$$\times \Bigg[\frac{C'^2}{\gamma_{m'n'} - \mu} \Bigg\{ \frac{\exp(-\mu(t_2 - t_1)) - \exp(-\mu t_2 - \gamma_{mn} t_1)}{\mu + \gamma_{mn}}$$

$$- \frac{\exp(-\gamma_{m'n'}(t_2 - t_1)) - \exp(-\gamma_{mn} t_1 - \gamma_{m'n'} t_2)}{\gamma_{mn} + \gamma_{m'n'}} \Bigg\}$$

$$+ \frac{C'^2}{\gamma_{mn} - \mu} \Bigg\{ \frac{\exp(-\gamma_{m'n'}(t_2 - t_1)) - \exp(-\mu t_1 - \gamma_{m'n'} t_2)}{\mu + \gamma_{m'n'}}$$

$$- \frac{\exp(-\gamma_{m'n'}(t_2 - t_1)) - \exp(-\gamma_{mn} t_1 - \gamma_{m'n'} t_2)}{\gamma_{mn} + \gamma_{m'n'}} \Bigg\}$$

$$+ \frac{1}{\gamma_{mn} + \gamma_{m'n'}} \Big\{ \exp(-\gamma_{m'n'}(t_2 - t_1)) - \exp(-\gamma_{mn} t_1 - \gamma_{m'n'} t_2) \Big\} \Bigg]$$

$$\ldots(14)$$

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References

Boley, B. A., and Weiner, J. H. (1960). Theory of Thermal Stresses. John Wiley & Sons, New York.

Nowacki, W. (1962). Thermoelasticity. Pergamon Press, London.

Roy Choudhuri, S. K. (1973). A note on quasi-static thermal deflection of a thin clamped circular plate due to ramp-type heating of a concentric circular region of the upper face. *J. Franklin Inst.*, **296**, No. 3, 213-19.

Srinivasan, S. K., and Mathews, P. M. (1956). Ordinary linear differential equations involving random functions. *Proc. Ind. Acad. Sci.*, **43**, 4-20.

Srinivasan, S. K., Subramanian, R., and Kumaraswamy, S. (1969). Unsteady motion of a viscous fluid subjected to random pulses. *Arch. Mech. Stos.* (2), **21**, 191.