GONČAROV POLYNOMIALS GENERATING FUNCTIONS OF TWO COMPLEX VARIABLES*

by M. NASSIF, *Department of Mathematics, Faculty of Science, University of Lagos, Yaba, Lagos, Nigeria*

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Two main results are established in the present paper concerning the Gončarov set $\{G_j(z;\ a_0, \dots, a_{j-1})\ .\ G_k(w;\ b_0, \dots, b_{k-1})\}$ of polynomials generating functions of two complex variables. The first result defines the class of entire functions represented by the set when the points $(a_j;\ b_k)$ lie in the unit ball, while the second result gives the exact form of the Cannon function of the set when $a_j = b_j = a\tau^j;\ |\tau| < 1$.

1. INTRODUCTION

This paper is concerned with the representation of functions of two complex variables by Gončarov polynomials. For this aim we suppose that $(a_j)_0^\infty$, $(b_k)_0^\infty$ are sequences of given complex numbers and that

$$\{G_j(z;\, a_0, \dots, a_{j-1})\}_0^\infty, \{G_k(z;\, b_0, \dots, b_{k-1})\}_0^\infty$$

are the sets of Gončarov polynomials associated with the points (a_j) and (b_k) respectively**. Therefore we have

$$\frac{z^m}{m!}\frac{w^n}{n!} = \left\{ \sum_{j=0}^{m} \frac{a_j^{m-j}}{(m-j)!} G_j(z;\, a_0, \dots, a_{j-1}) \right\} \times \left\{ \sum_{k=0}^{n} \frac{b_k^{n-k}}{(n-k)!} G_k(w;\, b_0, \dots, b_{k-1}) \right\};\ (m, n \geqslant 0). \quad \dots(1.1)$$

Thus the simple composite set† $\{g_{j,k}(z;\, w)\}$, given by

$$g_{j,k}(z;\, w) = G_j(z;\, a_0, \dots, a_{j-1}) \times G_k(w;\, b_0, \dots, b_{k-1});\ (j, k \geqslant 0) \quad \dots(1.2)$$

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**An exhaustive study of Gončarov polynomials is to be found in Buckholz's paper (1970).

†For the essentials of basic and composite sets of polynomials of several complex variables, the reader is referred to Nassif (1971).

serves as a base for the class of functions, of the two complex variables z and w, which are regular at the origin (0, 0). In fact, if*

$$f(z; w) = \sum_{m,n=0}^{\infty} \frac{f^{(m,n)}(0, 0)}{m!\, n!} z^m w^n$$

is such a function, then using (1.1) and (1.2) and interchanging the order of summation, we get the Gončarov series

$$f(z; w) \sim \sum_{j,k=0}^{\infty} f^{(j,k)}(a_j; b_k)\, g_{j,k}(z; w). \qquad ...(1.3)$$

Concerning the convergence of the Gončarov series (1.3), two main problems are considered in the present paper. In the first problem we assume that the points $(a_j; b_k) \in \bar{U}$, where $\bar{U}$ is the closed unit ball in the space $\mathbb{C}^2$ of the complex variables z and w, defined by

$$\bar{U} = \{(z, w) : |z|^2 + |w|^2 \leqslant 1\}. \qquad ...(1.4)$$

The second problem deals with the case where

$$a_j = b_j = \alpha\tau^j;\ (j \geqslant 0),$$

where α and τ are given complex numbers.

2. Case of Unit Ball

To formulate the result governing the case where the points $(a_j; b_k) \in \bar{U}$, we denote, with Buckholz (1970, p. 195),

$$H_n = \max |G_n(0; z_0, \ldots, z_{n-1})| \qquad ...(2.1)$$

where the maximum is taken over all sequences $(z_k)_0^{n-1}$ whose terms lie in the unit disk $|z| \leqslant 1$. Buckholz [1970, formula (1.2)] has proved that

$$\lim_{n\to\infty} H_n^{1/n} = H = \sup_{1\leqslant n<\infty} H_n^{1/n} = 1/W \qquad ...(2.2)$$

where W is the Whittaker constant†. With this notation, the following theorem is established.

Theorem 2.1 — When the points $(a_j; b_k) \in \bar{U}$, the Gončarov series (1.3) represents in any finite ball, every entire function $f(z; w)$ for which

*In the subsequent formula we adopt the notation that $f^{(j,k)}(a; b) = [\partial^{j+k} f(z; w)/\partial z^j \partial w^k]_{z=a,\, w=b}$.

†For definition of the Whittaker constant W (cf. Buckholz 1970, p. 194).

$$\limsup_{n\to\infty} \frac{\log M[r]}{r} < W \qquad ...(2.3)$$

where

$$M[r] = \max_{|z|^2+|w|^2 \leqslant r^2} |f(z; w)| . \qquad ...(2.4)$$

PROOF : We first observe, from the definition (1.4) of $\tilde{U}$, that we may assume that

$$|a_j| \leqslant \rho, \ |b_k| \leqslant \rho' \ (j, k \geqslant 0) \qquad ...(2.5)$$

where ρ and ρ' are non-negative numbers satisfying

$$\rho^2 + \rho'^2 \leqslant 1. \qquad ...(2.6)$$

Therefore, the relations (2.1), (2.2) and (2.5) easily lead to the inequalities

$$\left.\begin{array}{l} |G_p(0; a_0, ..., a_{p-1})| \leqslant (\rho H)^p \\ |G_p(0; b_0, ..., b_{p-1})| \leqslant (\rho' H)^p \end{array} (p \geqslant 0). \right\} \qquad ...(2.7)$$

Now, applying the formula [Nassif 1971, (3.4)] for the Cannon sum $\Omega_{m,n}[r]$ of the set $\{g_{j,k}(z; w)\}$, we can derive the following relation from (1.1) :

$$\Omega_{m,n}[r] = \sigma_{m,n}\, m!\, n! \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{|a_j|^{m-j}\, |b_k|^{n-k}}{(m-j)!\, (n-k)!} C_{j,k}[r] \qquad ...(2.8)$$

where

$$\sigma_{m,n} = \begin{cases} \dfrac{(m+n)^{(m+n)/2}}{m^{m/2}\, n^{n/2}} & \text{when } m, n \geqslant 0 \\ 1 \text{ when } m \text{ or } n = 0 \end{cases} \qquad ...(2.9)$$

and

$$C_{j,k}[r] = \max_{|z|^2+|w|^2 \leqslant r^2} |g_{j,k}(z; w)| . \qquad ...(2.10)$$

Also, writing

$$A_j(r) = \max_{|z|=r} |G_j(z; a_0, ..., a_{j-1})| \ ; \ B_k(r) = \max_{|w|=r} |G_k(w; b_0, ..., b_{k-1})|$$

then (1.2) and (2.10) together imply that

$$C_{j,k}[r] \leqslant A_j(r)\, B_k(r). \qquad ...(2.11)$$

Hence, applying the formula of Buckholz [1970, (2.6)]

$$G_j(z; a_0, ..., a_{j-1}) = \sum_{p=0}^{j} \frac{z^p}{p!} G_{j-p}(0; a_p, ..., a_{j-1})$$

and (2.7) in (2.11) we can easily arrive at the inequality

$$C_{j,k}[r] \leqslant \sum_{\mu=0}^{j} \sum_{\nu=0}^{k} \frac{\rho^{\mu}\rho'^{\nu}H^{\mu+\nu}}{(j-\mu)!\,(k-\nu)!}\, r^{j+k-\mu-\nu}$$

which can be introduced in (2.8) to yield, by the aid of (2.5),

$$\Omega_{m,n}[r] \leqslant m!\,n!\,\exp(2r+\rho+\rho')\,\sigma_{m,n} \sum_{\mu=0}^{m} (\rho H)^{\mu} \sum_{\nu=0}^{n} (\rho' H)^{\nu}. \quad \ldots(2.12)$$

Finally, we observe from the Buckholz bounds for H (Buckholz 1970, p. 194); namely,

$$\frac{1}{0.7378} < H < \frac{1}{0.7299}$$

that, in view of (2.6), the variation of ρ and ρ' is governed by the following alternatives :

$$\rho H \leqslant 1, \text{ and } \rho' H \leqslant 1; \text{ or } \rho H > 1, \text{ and } \rho' H \leqslant 1. \quad \ldots(2.13)$$

Easy calculations, from (2.9) and (2.12), based on the alternatives (2.13), lead to the conclusion that the set $\{g_{j,k}(z;\, w)\}$ is of increase not exceeding order 1, type $H = 1/W$. The required result follows then by applying the result (Nassif 1971, Theorem 4) and its obvious extension to the type. Theorem 2.1 is, therefore, proved.

To show that the result of Theorem 2.1 is best possible we recall that Buckholz (1970, Theorem 2) established the existence of a function $\phi(z)$ with the following properties:

(i) There is a sequence $(a_j)_0^{\infty}$ with $|\, a_j \,| \leqslant 1$, such that

$$\phi^{(j)}(a_j) = 0 \ \ (j \geqslant 0). \quad \ldots(2.14)$$

(ii) If $\Phi(r) = \max\limits_{|z|=r} |\,\phi(z)\,|$, then

$$\limsup_{r\to\infty} \log \Phi(r)/r = W. \quad \ldots(2.15)$$

Now, we can consider $\phi(z)$ as a function $f(z;\, w)$ of the two complex variables z and w, where

$$f(z;\, w) = \phi(z)\, h(w) \ \text{ with } \ h(w) \equiv 1.$$

Then in the notation (2.4) we see that

$$M[r] = \max_{|z|=r} |\,\phi(z)\,| = \Phi(r)$$

and therefore (2.15) yields

$$\limsup_{r\to\infty} \frac{\log M[r]}{r} = W. \qquad \dots(2.16)$$

Moreover, we see from (2.14) that

$$f^{(j,k)}(a_j;\, 0) = \begin{cases} 0 \text{ when } k > 0 \\ \phi^{(j)}(a_j) = 0 \text{ when } k = 0. \end{cases}$$

Since $| a_j | \leqslant 1$ we infer that each derivative of f has a zero in the unit ball $\bar{U}$; and hence, with the relation (2.16), compared with (2.3), it can be concluded that the result of Theorem 2.1 is best possible.

3. The Second Problem

When $a_j = b_j = \alpha\tau^j$, $j \geqslant 0$, it is convenient to write

$$| \alpha | = a, \; | \tau | = b. \qquad \dots(3.1)$$

Applying the result (Nassif 1971, Theorem 6) and its obvious extension for type it can be verified that the results (Nassif 1958, Theorems 1, 2) concerning the Goncarov polynomials

$$\{G_n(z;\, \alpha, \alpha\tau, \dots, \alpha\tau^{n-1})\} \qquad \dots(3.2)$$

for $b \geqslant 1$, apply directly to the composite set $\{g_{j,k}(z;\, w)\}$, given by

$$g_{j,k}(z;\, w) = G_j(z;\, \alpha, \dots, \alpha\tau^{j-1} \times G_k(w;\, \alpha, \dots, \alpha\tau^{k-1}). \qquad \dots(3.3)$$

We are, therefore, particularly concerned with the case where $b < 1$ as it yields a result distinct from that of the single variable case. More precisely, when $b < 1$ it has been shown [Nassif 1965, Theorem 2] that the Goncarov set (3.2) is effective in $| z | \leqslant r$ if and only if $r \geqslant a$. Hence the result (Nassif 1971, Theorem 5) implies that the composite set (3.3) will not be effective in any ball. The positive result established in what follows gives an exact expression for the Cannon function $\Omega[r]$ of the set $\{g_{j,k}(z;\, w)\}$, defined as

$$\Omega[r] = \limsup_{m+n\to\infty} \{\Omega_{m,n}[r]\}^{1/(m+n)}. \qquad \dots(3.4)$$

This result is stated as follows.

Theorem 3.1 — When $b < 1$, the Cannon function of the set $\{g_{j,k}(z;\, w)\}$ will be

$$\left.\begin{aligned} &\Omega[R] = a\sqrt{2} \text{ when } R \leqslant a; \\ \text{and} \quad &\Omega[R] = (a^2 + R^2)^{1/2} \text{ when } R \geqslant a. \end{aligned}\right\} \qquad \dots(3.5)$$

PROOF: First of all, applying the double inequality [Nassif 1971, (5.6)] for the Cannon sum $\Omega_{m,n}[r]$, we obtain

$$\begin{aligned} \sigma_{m,n}\{\sup_{0\leqslant t\leqslant 1} \omega_m(rt)\, \omega_n(rt')\} &\leqslant \Omega_{m,n}[r] \\ &\leqslant (m+1)(n+1)\, \sigma_{m,n}\{\sup_{0\leqslant t\leqslant 1} \omega_m(rt)\, \omega_n(rt')\} \end{aligned} \qquad \text{...(3.6)}$$

where $t'^2 = 1 - t^2$, $\sigma_{m,n}$ is given by (2.9) and, in view of (1.1),

$$\omega_m(r) = \sum_{j=0}^{m} \frac{m!\,(ab^j)^{m-j}}{(m-j)!} \left\{ \max_{|z|=r} |\, G_j(z;\alpha,\ldots,\alpha\tau^{j-1}) \,| \right\}. \qquad \text{...(3.7)}$$

Also, the following inequalities can be derived from the known results (Nassif 1958, Theorem 2) and (3.7)

$$\left.\begin{aligned} &\omega_n(r) > a^n \qquad (r \geqslant 0;\, n \geqslant 0) \\ &\omega_n(r) < K(\lambda a)^n\ (0 \leqslant r \leqslant a;\, n \geqslant 0) \end{aligned}\right\} \qquad \text{...(3.8)}$$

where λ is any finite number greater than 1 and K is a positive finite number independent of n and r.

We now suppose that $R \leqslant a$; then applying the second inequality of (3.8) in the right hand side of (3.6) and using the definition (2.9) of the number $\sigma_{m,n}$, we obtain

$$\Omega[R] \leqslant \limsup_{m+n\to\infty} \{K^2(m+1)(n+1)\,\sigma_{m,n}(\lambda a)^{m+n}\}^{1/(m+n)} \leqslant \lambda a\sqrt{2}.$$

Since λ can be arbitrarily chosen near to 1, it can be inferred that

$$\Omega[R] \leqslant a\sqrt{2}\ \ (R \leqslant a). \qquad \text{...(3.9)}$$

On the other hand, the first inequality of (3.8) and the left-hand side of (3.6) together yield

$$\Omega[R] \geqslant a\sqrt{2} \qquad \text{...(3.10)}$$

and the first equation of (3.5) follows from (3.9) and (3.10).

The case where $R > a$ is then considered. From the known results (Nassif 1965, Theorem 2) and (3.7), the following inequality can be derived.

$$\omega_n(r) < K_1(n+1)(\lambda r)^n\ \ (a \leqslant r \leqslant R;\, n \geqslant 0) \qquad \text{...(3.11)}$$

where K_1 [$> K$ of (3.8)] is independent of n and r.

An upper bound for the expression $\sigma_{m,n}\omega_m(Rt)\,\omega_n(Rt')$ is obtained by application of the definition (2.9) of $\sigma_{m,n}$, and the second inequality of (3.8) and the inequality

(3.11). In the evaluation of the upper bound the following alternatives for the variation of t and t' are accounted for

$$0 \leqslant t,\ t' \leqslant a/R;\quad a/R < t,\ t' \leqslant 1;\quad 0 \leqslant t \leqslant a/R < t' \leqslant 1$$

and the following inequalities are obtained :

$$\sigma_{m,n}\omega_m(Rt)\,\omega_n(Rt') \leqslant \begin{cases} K^2\sigma_{m,n}(\lambda a)^{m+n} < K^2(\lambda a\sqrt{2})^{m+n} \text{ when } 0 \leqslant t,\ t' \leqslant a/R \\ K^2(m+1)(n+1)(\lambda R)^{m+n}\sigma_{m,n}t^m t'^n \\ \leqslant K^2(m+1)(n+1)(\lambda R)^{m+n} \text{ when } a/R \leqslant t,\ t' \leqslant 1, \\ K^2(n+1)\lambda^{m+n}\sigma_{m,n}a^m R^n \\ \leqslant K^2(n+1)\lambda^{m+n}(a^2+R^2)^{(m+n)/2};\text{ when } 0 \leqslant t \leqslant a/R < t' \leqslant 1. \end{cases}$$

It follows that

$$\sigma_{m,n}\{\sup_{0\leqslant t\leqslant 1} \omega_m(Rt)\,\omega_n(Rt')\} \leqslant K^2(m+1)(n+1)\lambda^{m+n}(a^2+R^2)^{(m+n)/2}$$

whence the right-hand side of (3.6) and (3.4) imply that

$$\Omega[R] \leqslant \lambda(a^2+R^2)^{1/2}.$$

Therefore by the choice of λ we conclude that

$$\Omega[R] \leqslant (a^2+R^2)^{1/2}. \qquad \ldots(3.12)$$

To get an inequality in the opposite direction we choose t so small that

$$0 \leqslant Rt \leqslant a < Rt' \leqslant R$$

so that the first inequality of (3.8) yields

$$\sigma_{m,n}\omega_m(Rt)\,\omega_n(Rt') \geqslant \sigma_{m,n}a^m(Rt')^n. \qquad \ldots(3.13)$$

Corresponding to the positive integer m we choose the positive integer $n = n(m)$, say, such that

$$n = 1 < m(Rt'/a)^2 \leqslant n$$

so that

$$n(m) = (Rt'/a)^2\, m + \gamma;\ 0 \leqslant \gamma < 1. \qquad \ldots(3.14)$$

Application of (3.13) and (3.14) in the left-hand side of (3.6) leads easily to the following inequality

$$\Omega[R] \geqslant \limsup_{m\to\infty}\left[\frac{\left(a^2+R^2t'^2+\dfrac{a^2\gamma}{m}\right)^{(m+n)/2}}{\left(1+\dfrac{a^2\gamma}{mR^2t'^2}\right)^{n(m)/2}}\right]^{1/(m+n(m))} = (a^2+Rt'^2)^{1/2}.$$

Finally, observing that t can be chosen arbitrarily small and consequently t' can be taken arbitrarily near to 1, it can be deduced that

$$\Omega[R] \geqslant (a^2 + R^2)^{1/2}. \qquad \text{...(3.15)}$$

The second equation of (3.5) is implied by (3.12) and (3.15) and Theorem 3.1 is, therefore, established.

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