LINEAR FRACTIONAL FUNCTIONAL COMPLEMENTARY PROGRAMMING WITH EXTREME POINT OPTIMIZATION

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This paper deals with the problem of optimizing the ratio of two linear functions subject to a set of linear constraints and the complementarity condition with the additional restriction that the optimal solution should also be an extreme point of a second convex polyhedron. The mathematical formulation of the problem is

Maximize:

$$Z = \frac{d.x + e.u + f.v + g}{\alpha.x + \beta.u + \gamma.v + \delta}$$

subject to

$$A.x + B.u + C.v = h \qquad \text{(I)}$$

$$u.v = 0 \qquad \text{(complementarity condition)}$$

and that the solution (x, u, v) must also be an extreme point of

$$D.x + E.u + F.v = k$$

$$x, u, v \geqslant 0.$$

We shall call problem (I) as "linear fractional functional complementary programming problem with extreme point optimization". This problem could be treated as "extreme point linear fractional functional programming problem with non-linear constraints" because of the constraint $u.v = 0$. An enumerative procedure for solving such type of problems was proposed by Swarup *et al.* (1973) with linear objective function. The main difference between problem (I) and that considered by Swarup *et al.* (1973) other than the objective function is that problem (I) contains non-linear constraints of the type $u.v = 0$ only as in Ibaraki (1971). Due to this constraint, called the complementarity condition, a systematic procedure for solving problem (I) is proposed which seems to be more efficient than that of Swarup *et al.* (1973). A numerical example is also provided.

Introduction

Recently, an extreme point linear fractional functional programming problem was attacked (Puri 1973, Puri and Swarup 1974) in which the ratio of two linear functions is optimized subject to a set of linear inequalities with the additional restriction that the optimum solution should be at an extreme point of another convex polyhedron formed by another set of linear constraints. The authors (Garg and Swarup 1977) have considered the complementary programming problem with linear fractional functional in which the ratio of two linear functions is optimized subject to a set of linear inequalities and the complementarity condition, variables are of course to be non-negative and a branch and bound method was proposed for its solution.

In this paper, the ratio of two linear functions is to be optimized subject to a set of linear inequalities and the complementarity condition with the additional restriction that the optimal solution should also be at an extreme point of another convex polyhedron formed by another set of linear constraints. The mathematical model of the problem is

Maximize :

$$Z = \frac{d.x + e.u + f.v + g}{\alpha.x + \beta.u + \gamma.v + \delta}$$

subject to

$$A.x + B.u + C.v = h$$

$$u.v = 0$$

and that the solution (x, u, v) must be an extreme point of

$$D.x + E.u + F.v = k$$

$$x, u, v \geqslant 0$$

where x, u, v are n, m, m dimensional vectors respectively; A, B, C and D, E, F are matrices of order $m \times n$, $m \times m$, $m \times m$ and $p \times n$, $p \times m$, $p \times m$ respectively; d, e, f and α, β, γ are constant vectors of dimensions n, m, m respectively and g, δ are scalars. An enumerative procedure for solving problem of type (I) with linear objective function was proposed by Swarup *et al.* (1973) using extreme point mathematical programming (Swarup *et al.* 1972b, 1973). In view of the nature of the constraints $u.v = 0$, present in problem (I), a cutting plane method for solving problem (I) has been proposed in this paper with the help of complementary programming which seems to be more efficient than that given by Swarup *et al.* (1973).

Theoretical Development

Consider a related problem (I.1) :

Maximize :

$$Z = \frac{d.x + e.u + f.v + g}{\alpha.x + \beta.u + \gamma.v + \delta}$$

subject to

$$L.x + M.u + N.v = l$$
$$u.v = 0$$
$$x, u, v \geqslant 0$$

where $L = \begin{bmatrix} A \\ D \end{bmatrix}$ is an $(m + p) \times n$ matrix; $M = \begin{bmatrix} B \\ E \end{bmatrix}$ is an $(m + p) \times m$ matrix; $N = \begin{bmatrix} C \\ F \end{bmatrix}$ is an $(m + p) \times m$ matrix and $l = \begin{bmatrix} h \\ k \end{bmatrix}$ is an $m + p$ dimensional vector. Since any solution is an extreme point of

$$D.x + E.u + F.v = k$$
$$x, u, v \geqslant 0$$

and all the extreme points are finite, problem (I) is bounded. However problem (I.1) may or may not be bounded. We shall assume without any loss of generality that problem (I.1) is bounded, since addition of a single constraint $Z \leqslant K$, where K is an arbitrary large positive number, will provide us with a bounded problem and K being sufficiently large positive number, none of the extreme points of problem (I) are excluded from problem (I.1).

Notations

Let

J = set of all non-zero columns of (D, E, F).

$t = (x, u, v)$

$J(t)$ = set of all the columns of J associated with non-zero variables only.

$S_1 = \{t : A.x + B.u + C.v = h$ and t is an extreme point of $D.x + E.u + F.v = k;\ x, u, v \geqslant 0\}$

$S_2 = \{t : (x, u, v)$ is an extreme point of $L.x + M.u + N.v = l;\ x, u, v \geqslant 0\}$

$S_3 = \{t : L.x + M.u + N.v = l;\ x, u, v \geqslant 0\}$

$T_1 = Y_1 = \{t_{11}, t_{12}, \ldots, t_{1k_1}\}$ is the set of all the optimal extreme point solutions of problem (I.1).

s_1 = value of the objective function corresponding to the optimal solution Y_1 of the problem (I.1).

We know (Puri and Swarup 1974, Swarup *et al.* 1972a) that $S_1 \subseteq S_2$, i.e., every extreme point of $D.x + E.u + F.v = k;\ x, u, v \geqslant 0$ which is feasible for

$A.x + B.u + C.v = h$ is also an extreme point of $L.x + M.u + N.v = l$; $x, u, v \geqslant 0$. Thus our procedure for solving problem (I) involves working with problem (I.1).

PROCEDURE

Step 1 : To start with, solve problem (I.1) without the complementarity condition which is an ordinary linear fractional functional programming problem (Swarup 1965a,b). Examine whether the complementarity condition, $u.v = 0$, is satisfied.

Step 2 : If $u.v = 0$, go to step 4. If not, go to step 3.

Step 3 : If the complementarity condition $u.v = 0$ is not satisfied, use the branch and bound method proposed by the authors (Garg and Swarup 1977) and ensure that $u.v = 0$. Let T_1 be the set of all optimal extreme point solutions of problem (I.1). Let s_1 be the value of the objective function corresponding to the solution T_1. Go to step 4.

Step 4 : If $u.v = 0$, examine whether T_1, the set of all optimal extrerme point solutions of problem (I.1), is also an extreme point of problem (I) i.e. $T_1 \cap S_1 \neq \phi$. For this we shall proceed as follows. Let the number of columns in $J(t)$ be p^*. If $p^* = p$, we declare that T_1, the set of optimal extreme point solutions of problem (I.1) is an extreme point solution of problem (I) i.e. $T_1 \cap S_1 \neq \phi$. Terminate here. If $p^* \neq p$, $T_1 \cap S_1 = \phi$. Go to step 5.

Step 5 : If $T_1 \cap S_1 = \phi$, consider the following problem (I.2) :

Maximize :

$$Z = \frac{d.x + e.u + f.v + g}{\alpha.x + \beta.u + \gamma.v + \delta}$$

subject to

$$\begin{aligned} L.x + M.u + N.v &= l \\ Z &\leqslant s_1 \\ u.v &= 0 \\ x, u, v &\geqslant 0. \end{aligned}$$

Solve problem (I.2) without the complementarity condition. Repeat steps 1–4 again. Since the extreme points of $D.x + E.u + F.v = k$; $x, u, v \geqslant 0$ are finite, therefore, the procedure will come to an end after a finite number of steps. Suppose the procedure terminates at the ith stage. Let T_i, the ith best extreme point solution of problem (I.1), be also an extreme point solution of problem (I) i.e. $T_1 \cap S_1 \neq \phi$. Let s_1 be the value of the objective function corresponding to the ith best extreme point solution of problem (I.1). Terminate.

Numerical Example

Consider the following problem (I) which is a "linear fractional functional complementary programming problem with extreme point optimization", namely;

Maximize :

$$Z = \frac{2x_1 + x_2}{4x_1 + x_2 + 1}$$

subject to

$$\begin{aligned} -2x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 5x_2 + x_4 &= 23 \\ 2x_1 + x_2 + x_5 &= 15 \\ x_1 . x_2 &= 0 \end{aligned}$$

and (x_1, x_2) is an extreme point of

$$\begin{aligned} -3x_1 + 2x_2 + x_6 &= 4 \\ x_1 + 4x_2 + x_7 &= 22 \\ 5x_1 + 4x_2 + x_8 &= 46 \\ x_1 - 2x_2 + x_9 &= 5 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$

where $x_3, \ldots, x_9$ are slack variables. Consider the related problem (I.1) :

Maximize :

$$Z = \frac{2x_1 + 4x_2}{4x_1 + x_2 + 1}$$

subject to

$$\begin{aligned} -2x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 5x_2 + x_4 &= 23 \\ 2x_1 + x_2 + x_5 &= 15 \\ -3x_1 + 2x_2 + x_6 &= 4 \\ x_1 + 4x_2 + x_7 &= 22 \\ 5x_1 + 4x_2 + x_8 &= 46 \\ x_1 - 2x_2 + x_9 &= 5 \\ x_1 . x_2 &= 0 \\ x_1, x_2 &\geqslant 0. \end{aligned}$$

Solving problem (I.1) without the complementarity condition $x_1 . x_2 = 0$ which is an ordinary linear fractional functional programming problem, we get $x_1 = 3/2$, $x_2 = 4$,

$x_3 = 0,\ x_4 = 0,\ x_5 = 8,\ x_6 = 1/2,\ x_7 = 9/2,\ x_8 = 45/2,\ x_9 = 23/2;\ Z = 7/11$ (*see* Tableau 1). Since $x_1.x_2 \neq 0$, therefore, we shall make use of the method proposed by Garg and Swarup (1977).

Calculate the penalty functions $p(x_1)$ and $p(x_2)$ as follows :

$$\bar{u}_{13} = \frac{3/2 \times 3/2}{11} - \frac{5}{12} = -\frac{7}{13}$$

$$\bar{u}_{14} = \frac{3/2 \times -1/2}{11} + \frac{1}{12} = \frac{1}{66}$$

Since $\bar{u}_{14} > 0$ only, therefore

$$p(x_1) = \frac{3/2}{11} \times \frac{1}{11} \times \frac{1/6}{16/6} = \frac{3}{22}.$$

Similarly,

$$\bar{u}_{23} = 47/66,\ \bar{u}_{24} = -1/66.$$

Since $\bar{u}_{23} > 0$ only, therefore

$$p(x_2) = \frac{4}{11} \times \frac{1}{11} \times \frac{19/6}{47/66} = \frac{76}{177}.$$

Since $p(x_1) < p(x_2)$, choose the variable x_1 to be the departing variable from the basis. To derive the variable out of basis, a pivot operation is performed on the starred element of Tableau 1 (after that the column corresponding to the non-basic variable x_1 is deleted), we get

$$T_1 = \{x_1 = 0,\ x_2 = 1,\ x_3 = 0,\ x_4 = 18,\ x_5 = 14,\ x_6 = 2,\ x_7 = 18,\ x_8 = 42,\ x_9 = 7\},$$

the optimal extreme point solution to problem (I.1) with $s_1 = 7/11$, the value of the objective function corresponding to the optimal solution T_1.

Now examine whether T_1, the optimal extreme point solution of problem (I.1) is also an extreme point solution of problem (I). To do this we proceed as follows. Since

$$J(x, u, v) = \{x_2, x_6, x_7, x_8, x_9\} \text{ i.e. } |J(x, u, v)| = 5 > 4 = p.$$

Thus T_1, the optimal extreme point solution to problem (I.1) is not an extreme point solution of problem (I) i.e $T_1 \cap S_1 = \phi$. Consider problem (I.2) which is the same problem as (I.1) with the additional constraint $Z \leqslant 1/2$ i.e.

$$\frac{2x_1 + x_2}{4x_1 + x_2 + 1} \leqslant 1/2.$$

Solving problem (I.2) without the complementarity condition, we get

$$x_1 = 7,\ x_2 = 1,\ x_3 = 14,\ x_4 = 4,\ x_5 = 0,\ x_6 = 23,\ x_7 = 11,\ x_8 = 7,\ x_9 = 0,\ x_{10} = 0;\ Z = 1/2.$$

TABLEAU 1

			c_j	2	1	0	0	0	0	0	0	0
			d_j	4	1	0	0	0	0	0	0	0
D_H	C_H	Variables in basis	x_H	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
4	2	x_1	3/2	1	0	−5/12	1/12*	0	0	0	0	0
1	1	x_2	4	0	1	1/6	1/6	0	0	0	0	0
0	0	x_5	8	0	0	2/3	−1/3	1	0	0	0	0
0	0	x_6	1/2	0	0	−19/12	−1/12	0	1	0	0	0
0	0	x_7	9/2	0	0	−1/4	−3/4	0	0	1	0	0
0	0	x_8	45/2	0	0	17/12	−13/12	0	0	0	1	0
0	0	x_9	23/2	0	0	3/4	1/4	0	0	0	0	1
$z^{(2)} = 11$	$z^{(1)} = 7$	$c_j - z_j^{(1)}$		0	0	2/3	−1/3	0	0	0	0	0
$Z = 7/11$		$d_j - z_j^{(2)}$		0	0	3/2	−1/2	0	0	0	0	0
		Δ_j		0	0	−19/6	−1/6	0	0	0	0	0

Since $x_1.x_2 \neq 0$, therefore, again using the method proposed by Garg and Swarup (1977) we get

$$T_2 = \{x_1 = 5,\ x_2 = 0,\ x_3 = 11,\ x_4 = 13,\ x_5 = 5,\ x_6 = 19,\ x_7 = 17,\ x_8 = 21,\ x_9 = 0,\ x_{10} = 1\},$$

the optimal extreme point solution to problem (I.2) which is second best extreme point solution to problem (I.1) with $s_2 = 10/21$, the corresponding value of the objective function. Now examine whether T_2, the second best extreme point solution to problem (I.1) is also an extreme point solution to problem (I). Since

$$J(x, u, v) = \{x_1, x_6, x_7, x_8\} \text{ i.e. } |J(x, u, v)| = 4 = p.$$

Thus we declare that T_2, the second best extreme point solution of problem (I.1) is also an extreme point solution of problem (I) i.e. $T_2 \cap S_1 \neq \phi$. Terminate the procedure here. The optimal solution to problem (I) is given by

$$x_1 = 5,\ x_2 = 0;\ Z = 10/21.$$

Remarks

1. This process converges after a finite number of steps.

2. Note that in the above example, if one started from the optimal extreme point solution of problem (I.1) by using the extreme point mathematical programming technique (Puri and Swarup 1974) the fourth best extreme point solution to problem (I.1) would turn out to be the optimal solution of problem (I) thus requiring at least four steps. But with the help of the technique developed in this paper, the optimal solution to problem (I) is achieved in two steps only.

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