SEMI-SYMMETRIC METRIC CONNECTIONS IN AN ALMOST CONTACT MANIFOLD

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In the present paper we study the properties of curvature tensors following Sharfuddin and Hussain (1976) and define semi-symmetric T-connections. Some properties of almost Sasakian and Grayan manifolds vis-a-vis the Riemannian connections are also studied.

1. INTRODUCTION

Yano (1970) studied semi-symmetric metric connections in a Riemannian manifold and Mishra (1972) studied affine connections in an almost Grayan manifold. Sharfuddin and Hussain (1976) studied semi-symmetric metric connections in the setting of almost contact manifolds, particularly for almost Grayan manifolds and almost Sasakian manifolds. Following Sharfuddin and Hussain (1976) we study the properties of curvature tensors. We define semi-symmetric metric T-connections and study some properties of almost Sasakian and Grayan manifolds vis-a-vis the Riemannian connections.

Let V_n be an almost contact manifold with the structure tensor F, a vector field T and a contact form A such that

$$\text{(a)} \quad \bar{\bar{X}} = -X + A(X)\,T, \quad \text{(b)} \quad \bar{X} \overset{def}{=} F(X) \qquad \dots(1.1)$$

for arbitrary vector field X.

$$\text{(a)} \quad A(T) = 1, \quad \text{(b)} \quad A(\bar{X}) = 0, \quad \text{(c)} \quad \bar{T} = 0, \quad \text{Rank}\,(F) = n-1. \qquad \dots(1.2)$$

Let there be endowed in V_n a Riemannian metric tensor g such that

$$\left.\begin{array}{l} \text{(a)} \quad g(\bar{X}, \bar{Y}) = -\,g(\bar{\bar{X}}, Y), \quad \text{(b)} \quad g(X, T) = A(X) \\ \text{and evidently} \\ \text{(c)} \quad g(\bar{X}, \bar{Y}) = g(X, Y) - A(X)\,A(Y). \end{array}\right\} \qquad \dots(1.3)$$

Then V_n is called an almost Grayan manifold with $\{F, T, A, g\}$ structure. Putting

$$'F(X, Y) = g(\bar{X}, Y) \qquad \dots(1.4)$$

we have

$$\left.\begin{aligned} &\text{(a)} \quad 'F(\bar{X}, \bar{Y}) = 'F(X, Y) \\ &\text{(b)} \quad 'F(X, Y) + 'F(Y, X) = 0. \end{aligned}\right\} \qquad \ldots(1.5)$$

If D be a Riemannian connection in V_n, then Mishra (1972) has shown that

$$\left.\begin{aligned} &\text{(a)} \quad (D_X\, 'F)(\bar{Y}, \bar{\bar{Z}}) = (D_X\, 'F)(\bar{\bar{Y}}, \bar{Z}) \\ &\text{(b)} \quad (D_X\, 'F)(\bar{Y}, \bar{Z}) + (D_X\, 'F)(\bar{\bar{Y}}, \bar{\bar{Z}}) = 0 \end{aligned}\right\} \qquad \ldots(1.6)$$

and further, in an almost Grayan manifold the Nijenhuis tensor N satisfies

$$N(X, Y) = (D_{\bar{X}} F)(Y) - (D_{\bar{Y}} F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)}. \quad \ldots(1.7)$$

If in an almost Grayan manifold the 2-form $'F$ satisfies

$$2\, 'F(X, Y) = (D_X A)(Y) - (D_Y A)(X) = (dA)(X, Y) \qquad \ldots(1.8)$$

then V_n is called an almost Sasakian manifold and for a Sasakian manifold, we have from Mishra (1972)

$$(D_X\, 'F)(Y, Z) + (D_Y\, 'F)(Z, X) + (D_Z\, 'F)(X, Y) = 0. \qquad \ldots(1.9)$$

An almost Sasakian manifold is said to be Sasakian if T is a killing vector field, i.e., if

$$(D_X A)(Y) + (D_Y A)(X) = 0. \qquad \ldots(1.10)$$

A linear connection ∇ in V_n whose torsion tensor S satisfies

$$S(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = p(Y) X - p(X) Y \qquad \ldots(1.11)$$

where p is a 1-form, is called a semi-symmetric connection. Further, if $(\nabla_X g)(Y, Z) = 0$, then ∇ is called a semi-symmetric metric connection (Yano 1970).

2. Semi-Symmetric Metric T-Connection in Almost Contact Manifolds

In an almost contact manifold V_n a semi-symmetric connection ∇ is given by identifying the 1-form p of (1.11) with A of (1.1);

$$S(X, Y) = A(Y) X - A(X) Y. \qquad \ldots(2.1)$$

Let ∇' be a Riemannian connection and ∇ be a semi-symmetric metric connection in an almost Grayan manifold V_n. Let

$$\nabla_X Y = \nabla'_X Y + H(X, Y). \qquad \ldots(2.2)$$

Then it can be seen that (Yano 1970)

$$H(X, Y) = \tfrac{1}{2}\,[S(X, Y) + S'(X, Y) + S'(Y, X)]$$

where

$$g(S'(X, Y), Z) = g(S(Z, X), Y) \text{ and } S'(X, Y) = A(X)\, Y - g(X, Y)\, T.$$

Then we have

$$H(X, Y) = A(Y)\, X - g(X, Y)\, T \qquad ...(2.3a)$$

or

$$\nabla_X Y = \nabla'_X Y + A(Y)\, X - g(X, Y)\, T. \qquad ...(2.3b)$$

Now, if in addition

$$\text{(a)} \quad \nabla_X T = 0, \quad \text{or} \quad \text{(b)} \quad (\nabla_X A)(Y) = 0 \qquad ...(2.4)$$

then the connection ∇ is said to be a semi-symmetric metric T-connection. Thus from (2.3), we have

$$\left.\begin{array}{ll} \text{(a)} & \nabla'_X T + X - A(X)\, T = 0 \\ \text{or} & \\ \text{(b)} & (\nabla'_X A)(Y) + g(\bar{X}, \bar{Y}) = 0. \end{array}\right\} \qquad ...(2.5)$$

Theorem 2.1 — In an almost Sasakian manifold V_n with a semi-symmetric metric T-connection (2.3), we have

$$(dA)(X, Y) = 0 \qquad ...(2.6a)$$

$$(\nabla_X {'F})(Y, Z) = (\nabla'_X {'F})(Y, Z) \qquad ...(2.6b)$$

$$'H(X, T, Y) = {'H}(Y, T, X) \qquad ...(2.6c)$$

$$'N(X, Y, Z) = -2(\nabla'_Z {'F})(X, \bar{Y}) \qquad ...(2.6d)$$

$$(\nabla'_Y {'F})(X, T) = (\nabla_Y {'F})(X, T) = 0 \qquad ...(2.6e)$$

$$'N(T, Y, Z) = 0 = {'N}(X, T, Z). \qquad ...(2.6f)$$

PROOF : From eqn. (2.5*b*), we have

$$(\nabla'_X A)(Y) - (\nabla'_Y A)(X) = 0 = (dA)(X, Y)$$

which is (*a*). Differentiating $'F(Y, Z)$ covariantly with respect to ∇ and ∇' and using (2.3), we get

$$(\nabla_X {'F})(Y, Z) = (\nabla'_X {'F})(Y, Z) + A(Y)\, {'F}(Z, X) + A(Z)\, {'F}(X, Y) \qquad ...(2.7)$$

But, in consequence of eqn. (1.8), (2.7) gives (2.6*b*). Equation (2.6*c*) follows from (2.3), (2.6*a*) and the following equation

$$'H(X, Y, Z) = g(H(X, Y), Z).$$

Equation (2.6e) follows from taking covariant derivative of $'F(Y, T) = 0$ and using (2.5a). In consequence of eqns. (1.6a), (1.9) and (2.6e), eqn. (2.6d) follows, for

$$'N(X, Y, Z) \overset{def}{=} g(N(X, Y), Z).$$

Putting T for X and Y separately in (2.6d) and using (2.6e) we get (2.6f).

Hence the theorem follows.

Corollary 2.1 — A Sasakian manifold does not admit semi-symmetric metric T-connection.

PROOF : In a Sasakian manifold T is a Killing vector field, i.e.

$$(\nabla'_X A)(Y) + (\nabla'_Y A)(X) = 0$$

and from (2.5b) $(\nabla'_X A)(Y) - (\nabla'_Y A)(X) = 0$. These two equations give us

$(\nabla'_X A)(Y) = 0$, which implies that $g(\bar{X}, \bar{Y}) = 0$ from (2.5b) which is a contradiction.

Theorem 2.2 — Let V_n be an almost Grayan manifold admitting a semi-symmetric metric F connection (2.3). If the curvature tensor with respect to ∇ vanishes, then the manifold is projectively flat.

PROOF : Let

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

be the curvature tensor with respect to the semi-symmetric metric T-connection ∇ given by (2.3). In consequence of the eqns. (2.3b), (2.5b) and by simple calculation we get

$$R(X, Y, Z) = K(X, Y, Z) + g(Y, Z)\, X - g(X, Z)\, Y \qquad \text{...(2.8)}$$

where $K(X, Y, Z)$ is Christoffel-Riemann curvature tensor. According to the statement putting $R(X, Y, Z) = 0$ in (2.8) we have

$$K(X, Y, Z) = g(X, Z)\, Y - g(Y, Z)\, X \qquad \text{...(2.9)}$$

Contracting the above with respect to X, we get

$$\text{Ric}\,(Y, Z) = -\,(n - 1)\, g(Y, Z) \qquad \text{...(2.10)}$$

Putting (2.10) in (2.8), we get

$$0 = K(X, Y, Z) - \frac{1}{n-1}\{\text{Ric}\,(Y, Z)\, X - \text{Ric}\,(X, Z)\, Y\} \qquad \text{...(2.11)}$$

which implies that $P(X, Y, Z) = 0$, that is, projective curvature tensor vanishes.

Corollary (2.2) — In an almost Grayan manifold V_n, if the curvature tensor with respect to the semi-symmetric metric T-connection (2.3) vanishes, the following hold.

$$K(X, Y, T) + S(X, Y) = 0 \quad \text{...(2.12a)}$$

$$K(\bar{X}, \bar{Y}, T) = 0 \quad \text{...(2.12b)}$$

$$K(T, Y, \bar{Z}) = 'F(Y, Z)\, T \quad \text{...(2.12c)}$$

$$\bar{K}(T, Y, \bar{Z}) = 0 \quad \text{...(2.12d)}$$

$$'K(T, X, T, Y) + 'S(T, X, Y) = 0 = 'H(X, Y, T) + 'K(T, X, T, Y) \quad \text{...(2.12e)}$$

$$'K(T, X, T, Y) = g(\bar{X}, \bar{Y}) \quad \text{...(2.12f)}$$

$$'S(\bar{X}, Y, Z) + 'K(\bar{X}, Y, T, Z) = 0 \quad \text{...(2.12g)}$$

$$'K(\bar{X}, Y, T, Z) = 'H(X, Y, \bar{Z}) = -A(Y)\, 'F(X, Z) \quad \text{...(2.12h)}$$

where

$$\left.\begin{aligned} 'S(X, Y, Z) &\overset{def}{=} g(S(X, Y), Z) \\ 'H(X, Y, Z) &\overset{def}{=} g(H(X, Y), Z) \end{aligned}\right\} \quad \text{...(2.13a)}$$

$$'K(X, Y, Z, U) \overset{def}{=} g(K(X, Y, Z), U). \quad \text{...(2.13b)}$$

Proofs of the above relations follow by simple calculations using the properties obtained previously.

3. Semi-symmetric Metric (F, T)-Connections

Let ∇ be a linear connection in an almost contact manifold V_n. Let its torsion tensor be given by

$$S(X, Y) = A(Y)\, X - A(X)\, Y - 2\, 'F(X, Y)\, T = \nabla_X Y - \nabla_Y X - [X, Y]. \quad \text{...(3.1)}$$

Let ∇' be a Riemannian connection in V_n, then we have, in consequence of Yano (1970),

$$H(X, Y) = A(Y)\, X + A(Y)\, \bar{X} - g(X, Y)\, T - 'F(X, Y)\, T + A(X)\, \bar{Y} \quad \text{...(3.2)}$$

or

$$\nabla_X Y = \nabla'_X Y + A(Y)\,(X + \bar{X}) - (g(X, Y) + 'F(X, Y))\, T + A(X)\, \bar{Y}. \quad \text{...(3.3)}$$

The given connection ∇ be a metric (F, T)-connection iff (Yano 1970)

$$\text{(a) } \nabla_X g = 0, \quad \text{(b) } \nabla_X F = 0 \text{ and } \quad \text{(c) } \nabla_X A = 0, \text{ or } \quad \text{(d) } \nabla_X T = 0. \qquad \ldots(3.4)$$

In consequence of (3.4*b*) and (3.4*c*) we have

$$(\nabla'_X \,'F)(Y, Z) = A(Y)('F(X, Z) - g(X, Z)) - A(Z)(F'(X, Y) - g(X, Y)) \qquad \ldots(3.5)$$

$$(\nabla'_X \,'F)(\bar{Y}, \bar{Z}) = 0. \qquad \ldots(3.6)$$

$$\bar{\bar{X}} = \nabla'_X T + \bar{X} \qquad \ldots(3.7)$$

$$(\nabla'_X A)(Y) = \,'F(Y, X) - g(\bar{X}, \bar{Y}) \qquad \ldots(3.8)$$

From (3.1), it is easily seen that

$$\text{(a) } \overline{S(\bar{X}, \bar{Y})} = 0, \quad \text{(b) } S(X, T) = X - A(X)\, T, \quad \text{(c) } S(\bar{X}, T) = \bar{X},$$
$$\text{(d) } S(\bar{\bar{X}}, T) + S(X, T) = 0, \quad \text{(e) } A(S(\bar{X}, T)) = 0. \qquad \ldots(3.9)$$

Now, let us put

$$\text{(a) } g(H(X, Y), Z) = \,'H(X, Y, Z), \quad \text{(b) } g(S(X, Y), Z) = \,'S(X, Y, Z) \qquad \ldots(3.10)$$

then we can write

$$\text{(a) } 'S(X, Y, Z) = A(Y)\, g(X, Z) - A(X)\, g(Y, Z) - 2A(Z)\,'F(X, Y)$$
$$\text{(b) } 'H(X, Y, Z) = A(Y)\,(g(X, Z) + \,'F(X, Z)) - A(Z)\,(g(X, Y) + \,'F(X, Y)) + \,'F(Y, Z)\, A(X). \qquad \ldots(3.11)$$

Theorem 3.1 — In an almost Grayan manifold with a semi-symmetic metric $(F - T)$-connection (3.3) and (3.4), the following results hold :

(i) $'S(X, \bar{Y}, \bar{Z}) + \,'H(X, \bar{Y}, \bar{Z}) = 0$

(ii) $'S(\bar{X}, Y, \bar{Z}) = \,'H(\bar{X}, Y, \bar{Z}) + A(Y)\, F(Z, X)$,

(iii) $'S(X, Y, Z) + \,'S(Y, Z, X) + \,'S(Z, X, Y) = 2[\,'H(X, Y, Z) + \,'H(Y, Z, X) + \,'H(Z, X, Y)]$

$= \,'H(\bar{X}, Y, Z) + \,'H(\bar{Y}, Z, X) + \,'H(\bar{Z}, X, Y) = 2\,[A(Y)\,'F(X, Z) + A(Z)\,'F(Y, X) + A(X)\,'F(Z, Y)]$,

(iv) $'H(\bar{X}, Y, \bar{Z}) + H(\bar{Z}, Y, \bar{X}) = 2\,'H(Y, \bar{X}, \bar{Z})$,

(v) $'H(\bar{X}, Y, Z) + \,'H(X, \bar{Y}, Z) + \,'H(X, Y, \bar{Z}) = 0$,

(vi) $'S(\bar{X}, Y, Z) + 'S(X, \bar{Y}, Z) + 'S(X, Y, \bar{Z}) = 0,$

(vii) $'H(\bar{\bar{X}}, Y, Z) + 'H(X, Y, Z) = A(X)\,'F(Y, Z),$

(viii) $'S(X, Y, \bar{\bar{Z}}) + 'S(X, Y, Z) = 2A(Z)\,'F(X, Y),$

(ix) $2\,[H(\bar{\bar{X}}, Y, Z) + 'H(X, Y, Z)] + 'S(Y, Z, \bar{\bar{X}}) + 'S(Y, Z, X) = 0.$ ...(3.12)

The proofs of these results follow by the simple calculations and use of the properties obtained before.

Corollary 3.1 — In an almost Sasakian manifold with semi-symmetric metric $(F - T)$-connection (3.3), we have

$$'S(K, Y, Z) + 'S(Y, Z, X) + 'S(Z, X, Y) = 0 \quad ...(3.13a)$$

$$'H(X, Y, Z) + 'H(Y, Z, X) + 'H(Z, X, Y) = 0 \quad ...(3.13b)$$

$$'H(\bar{X}, Y, Z) + 'H(\bar{Y}, Z, X) + 'H(\bar{Z}, X, Y) = 0 \quad ...(3.13c)$$

$$'S(\bar{X}, Y, \bar{Z}) + 'S(\bar{Y}, Z, \bar{X}) + 'S(\bar{Z}, X, \bar{Y}) = 'H(\bar{X}, Y, \bar{Z}) + 'H(\bar{Y}, Z, \bar{X}) + 'H(\bar{Z}, X, \bar{Y}) \quad ...(3.13d)$$

$$H(X, \bar{Z}, \bar{Y}) + 'H(Y, \bar{X}, \bar{Z}) + 'H(Z, \bar{Y}, \bar{X}) = 0. \quad ...(3.13e)$$

The proof follows from eqns. (1.9), (3.5) and Theorem 3.1.

Let $N(X, Y)$ be the Nijenhuis tensor given by

$$N(X, Y) = (\nabla'_{\bar{X}} F)(Y) - (\nabla'_{\bar{Y}} F)(X) - (\overline{\nabla'_X F)(Y}) + (\overline{\nabla'_Y F)(X}). \quad ...(3.14)$$

Then, in consequence of eqn. (3.5), we have

$$'N(X, Y, Z) = 2A(Z)\,'F(X, Y) \quad ...(3.15a)$$

where

$$'N(X, Y, Z) \overset{def}{=} g(N(X, Y), Z). \quad ...(3.15b)$$

Theorem 3.2 — In an almost Grayan manifold with a semi-symmetric metric (F, T)-connection (3.3), we have

(i) $(N(X, T, Z) = 'N(T, Y, Z) = 0,$

(ii) $'N(X, Y, \bar{Z}) = 0,$

(iii) $2\,'H(X, \bar{Y}, \bar{Z}) = 'N(\bar{Y}, \bar{Z}, X) = 'N(Y, Z, X),$

(iv) $2\,'S(X, \bar{Y}, \bar{Z}) = 'N(\bar{Y}, Z, X),$

(v) $'S(\bar{X}, \bar{Y}, Z) + 'N(\bar{X}, \bar{Y}, Z) = 0,$

(vi) $'H(X, Y, Z) - 'S(Z, Y, X) = \frac{1}{2}['N(Z, Y, X) + 'N(X, Z, Y) + 'N(Y, X, Z)],$

(vii) $'H(\bar{\bar{X}}, Y, Z) + 'H(X, Y, Z) = \frac{1}{2}\,'N(Y, Z, X),$

(viii) $'S(X, Y, \bar{\bar{Z}}) + 'S(X, Y, Z) = 'N(X, Y, Z).$ ...(3.16)

Proofs of all the above results follow from eqns. (3.15*a*), (3.11*a*), (3.11*b*) and Theorem 3.1.

Corollary 3.2 — In an almost Sasakian manifold with a semi-symmetric metric (F, T)-connection (3.3), we have

$$'N(X, Y, Z) + 'N(Y, Z, X) + 'N(Z, X, Y) = 0. \quad ...(3.17)$$

The proof is obvious.

Now, let $R(X, Y, Z)$ be the curvature tensor with respect to the semi-symmetric metric (F, T)-connection (3.3) and $K(X, Y, Z)$ be the Riemann-Christoffel curvature tensor in an almost Grayan manifold. In consequence of (3.3) we can easily obtain

$$\begin{aligned} R(X, Y, Z) = {} & K(X, Y, Z) + (g(Y, Z) + 'F(Y, Z)\,(X + \bar{X}) - (g(X, Z) \\ & + 'F(X, Z))\,(Y + \bar{Y}) + A(Z)\,A(Y)\,\bar{X} - A(Z)\,A(X)\,\bar{Y} \\ & - A(Z)\,A(Y)\,X + A(Z)\,A(X)\,Y - A(Y)\,T\,'F(X, Z) \\ & + A(X)\,T\,'F(Y, Z) + A(Y)\,T\,g(X, Z) - A(X)\,T\,g(Y, Z) \\ & - 2\bar{Z}\,'F(X, Y). \end{aligned} \quad ...(3.18)$$

Theorem 3.3 — Let V_n be an almost Grayan manifold endowed with a semi-symmetric metric (F, T)-connection (3.3). If the curvature tensor with respect to this connection vanishes, then we have

$$K(X, Y, T) = 2\,[A(X)\,\bar{Y} - A(Y)\,\bar{X}] \quad ...(3.19a)$$

$$\bar{K}(X, Y, T) = 2\,[A(Y)\,X - A(X)\,Y] \quad ...(3.19b)$$

$$A(K(X, Y, T)) = 0 \quad ...(3.19c)$$

$$A(K(T, Y, Z)) + 2\,'F(Y, Z) = 0 \quad ...(3.19d)$$

$$\text{Ric}\,(Y, T) = 0. \quad ...(3.19e)$$

Proof of the theorem follows from the supposition $R(X, Y, Z) = 0$ and (3.18).

Proposition 3.1 — In a Sasakian manifold eqn. (3.3) reduces to

$$\nabla_X Y = \nabla'_X Y - A(Y)\,\bar{X} + A(X)\,\bar{Y} + 'F(X, Y)\,T. \quad ...(3.20)$$

PROOF : Putting $\nabla'_Y T = \bar{Y}$ in (3.3), (3.20) follows immediately.

The curvature tensor of the connection (3.20) is given by

$$R(X, Y, Z) = K(X, Y, Z) + \bar{X}\,'F(Y, Z) - \bar{Y}\,'F(X, Z) + A(Z)\,A(X)\,Y - A(Z)\,A(Y)\,X + A(Y)\,T\,g(X, Z) - A(X)\,T\,g(Y, Z) + 2\bar{Z}\,'F(X, Y). \quad \text{...(3.21)}$$

Theorem 3.4 — In a Sasakian manifold with semi-symmetric metric (F, T)-connection (3.20), we have

$$\left.\begin{aligned} &\text{(a)} \quad 'R(X, Y, Z, T) = 0 \\ &\text{(b)} \quad R(T, Y, Z) = 0 \\ &\text{(c)} \quad R(X, Y, T) = 0. \end{aligned}\right\} \quad \text{...(3.22)}$$

PROOF : In a Sasakian manifold V_n, we know that

$$\left.\begin{aligned} &\text{(a)} \quad (D_Z\,'F)(X, Y) = \,'K(X, Y, Z, T) = A(X)\,g(Y, Z) - A(Y)\,g(X, Z) \\ &\text{(b)} \quad K(T, Y, Z) = g(Y, Z)\,T - A(Z)\,Y \\ &\text{(c)} \quad K(X, Y, T) = A(Y)\,X - A(X)\,Y \\ &\text{(d)} \quad 'K(T, Y, Z, T) = g(\bar{Y}, \bar{Z}) = g(Y, Z) - A(Y)\,A(Z). \end{aligned}\right\} \quad \text{...(3.23)}$$

Thus, in consequence of eqns. (3.21) and (3.23), eqns. (3.22) follow.

Theorem 3.5 — In a Sasakian manifold, endowed with a semi-symmetric metric $(F - T)$-connection with respect to which the curvature tensor vanishes, we have

$$\left.\begin{aligned} &\text{(a)} \quad 'K(\bar{X}, Y, Z, T) + \,'K(X, \bar{Y}, Z, T) + \,'K(X, Y, \bar{Z}, T) = 0 \\ &\text{(b)} \quad (\nabla'_T\,'K)(X, Y, Z, T) = (\nabla_T\,'K)(X, Y, Z, T). \end{aligned}\right\} \quad \text{...(3.24)}$$

PROOF : In consequence of eqn. (3.21), we get for $R(X, Y, Z) = 0$,

$$K(\bar{X}, Y, Z) + K(X, \bar{Y}, Z) + K(X, Y, \bar{Z}) + \bar{\bar{X}}\,'F(Y, Z) + \bar{\bar{Y}}\,'F(Z, X) + 2\bar{\bar{Z}}\,'F(X, Y) = 0. \quad \text{...(3.25)}$$

We easily obtain (3.24*a*) from eqn. (3.25).

Now, we can calculate covariant derivative of the Riemannian curvature tensor with respect to ∇ and ∇' respectively and write down

$$(\nabla_T K)(X, Y, Z) = (\nabla'_T K)(X, Y, Z) - \{K(\bar{X}, Y, Z) + K(X, \bar{Y}, Z) + K(X, Y, \bar{Z})\} \quad \ldots(3.26)$$

which gives the eqn. (3.24*b*), in consequence of (3.24*a*).

Hence we proved the theorem.

§4. *Theorem* 4.1 — Let V_n be a Sasakian manifold endowed with a semi-symmetric connection (Sharfuddin and Hussain 1976). If the curvature tensor with respect to this connection vanishes, then the manifold is conformally flat.

PROOF : Let ∇ be a semi-symmetric metric connection in a Sasakian manifold given by

$$\left.\begin{array}{ll} \text{(a)} & S(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = A(Y)X - A(X)Y \\ \text{(b)} & \nabla_X Y = \nabla'_X Y + A(Y)X - g(X, Y)T, \end{array}\right\} \quad \ldots(4.1)$$

where ∇' is Riemannian connection. We can easily show

$$R(X, Y, Z) = K(X, Y, Z) - X'F(Y, Z) + 'F(X, Z)Y - g(Y, Z)X + g(X, Z)Y - g(Y, Z)\bar{X} + g(X, Z)\bar{Y} - A(Z)A(X)Y + A(Z)A(Y)X - A(Y)Tg(X, Z) + A(X)Tg(Y, Z). \quad \ldots(4.2)$$

Let $R(X, Y, Z) = 0$. Then contracting (4.2), we get

$$\left.\begin{array}{ll} \text{(a)} & \frac{1}{n-2}\,\text{Ric}\,(Y, Z) = 'F(Y, Z) - A(Z)A(Y) + g(Y, Z) \\ \text{(b)} & \frac{1}{n-2}\,r(Y) = \bar{Y} - A(Y)T + Y \\ \text{(c)} & R = (n-1)(n-2). \end{array}\right\} \quad \ldots(4.3)$$

Then putting these results in (4.2), we get

$$K(X, Y, Z) - \frac{1}{n-2}\{\text{Ric}\,(Y, Z)X - \text{Ric}\,(X, Z)Y - g(X, Z)r(Y) + g(Y, Z)r(X)\} + \frac{R}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\} = 0 \quad \ldots(4.4)$$

which proves the statement.

Corollary 4.1 — In a Sasakian manifold endowed with a semi-symmetric metric connection whose curvature tensor vanishes, we have

$$\left.\begin{aligned}
&\text{(a)} \quad K(X, Y, T) = A(Y)\,\bar{X} - A(X)\,\bar{Y} \\
&\text{(b)} \quad K(T, Y, Z) = {}'F(Y, Z) - A(Z)\,\bar{Y} \\
&\text{(c)} \quad {}'K(T, Y, Z, T) + K(T, Z, Y, T) = 0 \\
&\text{(d)} \quad {}'K(T, \bar{Y}, \bar{Z}, T) = {}'K(T, Y, Z, T) \\
&\text{(e)} \quad \bar{K}(X, Y, T) + S(X, Y) = 0.
\end{aligned}\right\} \qquad \ldots(4.5)$$

Proof of the Corollary follows from eqn. (4.2).

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