ON A CLASS OF LATTICE ORDERED NEAR-RINGS

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In this paper we continue the study of a class of lattice ordered near-rings, called f-near-rings, introduced in an earlier paper (Radhakrishna and Bhandari 1977). The concepts of l-primitivity, J_ν-radicals ($\nu = 0$ or 2) are introduced and structure theorems for J_ν-semisimple f-near-rings are given.

1. INTRODUCTION

Throughout this paper a near-ring is taken to satisfy the left distributive law and $0.a = 0$ for all a in the near-ring. It is assumed that reader is familiar with the basic properties of near-rings and we refer to Blackett (1953) and Johnson (1960). Pilz (1971) initiated the study of ordered near-rings. He defined an ordered near-ring as a near-ring which is a fully ordered group under addition and the product of two positive elements is positive. In such a case, unlike rings, product of two negative elements need not be positive. However this condition is necessary for establishing a structure theory for fully ordered (f.o.) and lattice ordered (l.o.) near-ring. In an earlier communication (Radhakrishna and Bhandari 1977), we have modified the definition of partially ordered (p.o.) near-rings as follows :

Definition 1.1 — A near-ring $(N, +, .)$ is said to be a p.o. near-ring under an ordering '$\leqslant$' if $(N, +, \leqslant)$ is a p.o. group and satisfies

(1) $a \leqslant b$, $c \geqslant 0$ implies $ca \leqslant cb$ and $ac \leqslant bc$.

A p.o. near-ring N is said to be fully ordered (lattice ordered) if $(N, +)$ is a fully ordered (lattice ordered) group. Concepts of l-ideals, N-l-subgroups 0-homomorphism are defined as for rings and homomorphism theorems hold (Johnson 1960). In a p.o. ring R the following three statements are equivalent (Fuchs 1963).

(2) For $a, b, c \in R$, $a \wedge b = 0$, $c \geqslant 0$ implies $ac \wedge b = 0 = ca \wedge b$,

(3) R is a subring and a sublattice of a complete direct sum of f.o. near-rings,

(4) For every subset X of R, $\bar{X}$ is an l-ideal where
$\bar{X} = \{y \in R \mid y \wedge x = 0 \text{ for all } x \in X\}$, called the polar of X.

Following the technique of Pierce (1956), it was shown by the authors that for l.o. near-rings (3) and (4) are equivalent but (2) does not imply (4) in general (Radhakrishna and Bhandari 1977). Any l.o. near-ring satisfying (3) [or equivalently (4)] is called an *f*-near-ring (*f*-*n*-ring). Another difference that can be noted between l.o. rings and l.o. near-rings is that the inequality ' $| ab | \leqslant | a | . | b |$ ' need not hold in the latter case. The following example shows that it need not be true even for f.o. near-rings.

Example 1 — Let $N = \{ax + bx^2 \mid a, b$ are real numbers$\}$. Define addition component-wise, order by

$$ax + bx^2 \geqslant 0 \text{ if } a > 0 \text{ or } a = 0 \text{ and } b \geqslant 0,$$

and multiplication '.' by

$$(ax + bx^2) . (cx + dx^2) = \begin{cases} ac\, x^2 & \text{if } a < 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that N is a f.o. near-ring and

$$| (-x) . (x + x^2) | > | -x | . | x + x^2 | .$$

In the rest of this paper we consider only those l.o. near-rings in which inequality $| ab | \leqslant | a | . | b |$ holds for all elements a and b.

Concept of l_p-group and l_p-ideals for l.o. near-rings are introduced in Section 2, which helps us to study l-primitivity and J_ν-radicals for *f*-*n*-rings in Section 3 without introducing module theory (Steinberg 1972). Properties of J_ν-radicals under chain conditions are given in Section 4.

2. l_p-Group and l_p-Ideal in *l.o.* near Rings

Definition 2.1 — Let $(N, +, .)$ be an l.o. near-ring. A convex l-subgroup H of $(N, +)$ is said to be an l_p-subgroup if $a, b \in N^+ \setminus H$ implies $a \wedge b \in N^+ \setminus H$, where N^+ denotes the set of positive elements in N.

An l_p-ideal of an l.o. near-ring is an l_p-subgroup which is also an l-ideal. A subset A of N^+ is said to be a lower sublattice if a, b in A implies $a \wedge b \in A$. It is easy to see that a convex l-subgroup H of an l.o. near-ring N is an l_p-subgroup if and only if $N^+ \setminus H$ is a lower sublattice of N. Moreover convex l-subgroups of N containing an l_p-subgroup form a chain.

Lemma 2.2 — If S is a lower sublattice of an l.o. near-ring N not containing zero, then there exists a convex l_p-subgroup H which is disjoint from S and is minimal in the sense that it cannot contain any proper l_p-subgroup.

Proof : By Zorn's lemma, S can be extended to a maximal lower sublattice S' not containing zero and hence there exists a maximal convex l-subgroup H of $(N, +)$

disjoint from S. It is easy to see that H is the required l_p-subgroup. An immediate consequence is the following.

Corollary 2.3 — Every minimal convex l_p-subgroup of an l.o. near-ring N is a union of polars.

The following theorem gives another characterization of *f-n*-rings.

Theorem 2.4 — An l.o. near-ring N is an *f-n*-ring if and only if each minimal convex l_p-subgroup is an l-ideal.

PROOF: By Corollary 2.2, it is clear that in an *f-n*-ring every minimal convex l_p-subgroup is an l-ideal. Conversely if for some a in N there is a $y > 0$ in N such that $y \notin \bar{a}$ (polar of a) then $S = \{ | a | , y, | a | \wedge y\}$ is a lower sublattice not containing zero. Hence by lemma there exists a minimal l_p-subgroup I_y disjoint from S. Certainly this contains $\bar{a}$. Thus $\bar{a}$, being the intersection of convex l_p-subgroups, is an l-ideal. So N is an *f-n*-ring.

In *f-n*-rings an l_p-ideal behaves in much the same way as a 'prime convex l-subgroup' in a l.o. group (Conrad 1965). Proof of the following theorem is similar to that for *f*-rings and hence omitted.

Theorem 2.5 — Let N be an *f-n*-ring.

(*i*) An l-ideal I of N is an l_p-ideal if and only if N/I is an f.o. near-ring.

(*ii*) If S is a lower sublattice of N not containing zero, then $N^+ \setminus s$ is an l_p-ideal disjoint from S.

Definition 2.6 — An l-ideal I of an *f-n*-ring is called a regular l-ideal if there exist an element $n > 0$ in N such that I is maximal with respect to the property of not containing n.

If I is a regular l-ideal of an *f-n*-ring N, then N/I is a subdirectly irreducible *f-n*-ring and hence N/I is fully ordered. Consequently I is an l_p-ideal.

3. J_ν-RADICAL ($\nu = 0$ OR 2) OF *f-n*-RINGS

A right l-ideal I of an *f-n*-ring is said to be modular if there is an $a > 0$ in N such that x-ex is in I for every x in N.

Definition 3.1 — A right l-ideal I of an *f-n*-ring N is said to be a modular right l-ideal of

(*i*) type two if I is a modular right l-ideal and there is no N-l-subgroup of N containing I properly.

(*ii*) type zero if I is a modular right l-ideal and is maximal as a right l-ideal. The concept of l-primitivity can be defined as for *f*-rings.

Definition 3.2 — An l-ideal P of an f-n-ring N is said to be l-primitive of type two (zero) if there exists a modular right l-ideal of type two (zero) such that

$$P = \{x \in N \mid Nx \subseteq I\} \equiv (I : N).$$

An f-n-ring N is said to be l-primitive of type two (zero) if (0) is an l-primitive l-ideal of type two (zero).

Lemma 3.3 — If I is a modular right l-ideal of an f-n-ring N of type two or zero, then I is a convex l_p-subgroup of N.

PROOF : It suffices to show that for $a > 0$, $b > 0$, a and b not in I, $a \wedge b \neq 0$. Suppose $a \wedge b = 0$, then $< a >$, the l-ideal generated by a, is contained in the polar of b. Since $a \in I$, $I + < a > \neq I$. Therefore, $N = I + < a >$ and $b = c + d$ for some $c \in I$ and $d \in < a >$. Since

$$b = b \wedge (c + d) \leqslant b \wedge (\mid c \mid + \mid d \mid + \mid c \mid) \leqslant 2(b \wedge \mid c \mid),$$

$b \in I$ which is not the case. Hence $a \wedge b \neq 0$.

If P is an l-primitive l-ideal of type two (zero), there is a modular right l-ideal of type two (zero) such that $P = (I : N)$ and P is the largest l-ideal of N contained in I. Since I is a convex l_p-subgroup, I contains a minimal convex l_p-subgroup, say H, of N. By Theorem 2.4 H is an l-ideal of N. So $H \subseteq P$. If P' is an l-ideal of N containing P, then P' and I both contain H and so they are comparable. If $P' \subseteq I$ then $P' = P$ and if $I \subseteq P'$, then $P' = N$. This proves the following

Corollary 3.4 — If P is an l-primitive l-ideal of type two or zero then P is an l_p-ideal and is a maximal l-ideal.

Corollary 3.5 — An l-primitive f-n-ring of type two or zero is a f.o. near-ring.

Theorem 3.6 — If I is a modular right l-ideal of type two, then I is an l-ideal of N.

PROOF : Let $P = (I : N)$. Then P is an l-primitive l-ideal of type two. If $P \neq I$, there exists an $a \in I$ such that $Na \not\subseteq I$. So for some $g > 0$ in N, $g\,a \notin I$. Let $K = \{x \in N \mid \mid x \mid \leqslant gy$ for some y in $I\}$. $K + P$ is an N-l-subgroup of N containing P properly. Since P is an l_p-ideal, the convex l-subgroups of (N, T) containing P form a chain. So either $K + P \subseteq I$ or $I \subseteq K + P$. If $K + P \subseteq I$ then $K \subseteq I$ which means $gI \subseteq I$, a contradiction. Therefore $I \subseteq K + P$ and hence $K + P = N$, as I is a modular right l-ideal of type two. If e is the left identity modulo I, $ge = c + d$, for some $c \in K$, $d \in P$. There exists $b \in I$ such that $\mid c \mid \leqslant gb$ and so $ge + P = c + P \leqslant gb + P$ in the f.o. near-ring N/P. But convexity of I implies that $e + P > x + P$ for every $x \in I$ and hence $ge + P = gb + P$, or $g(e - b) \in P$. Let $I' = \{x \in N \mid gx \in P\}$. Then I' is a right l-ideal containing P and so either $I \subseteq I'$ or $I' \subseteq I$. If $I \subseteq I'$ then $gI \subseteq P \subseteq I$, a contradiction. So $I' \subseteq I$, Then $e - b \in I$ and so $e \in I$, a contradiction. Hence $I = P$.

By taking $P = (0)$ we have the following :

Theorem 3.7 — An *f-n*-ring is *l*-primitive of (i) type two if and only if N is an f.o. near ring with left identity and without N-*l*-subgroups and (ii) type zero if and only if N is an f.o. near-ring with a unique maximal modular right *l*-ideal I such that $(I : N) = (0)$.

In an *f-n*-ring N, $J_\nu(N)$, the intersection of all *l*-primitive *l* ideal of type ν ($\nu = 2$ or 0) is called the $J\nu$-radical of N. If there are no *l*-primitive *l*-ideals of type ν, then $J_\nu(N) = N$. By Theorem 3.5 $J_2(N)$ is the intersection of all modular right *l*-ideals of type two. Moreover it can be verified that $J_\nu(N/J_\nu(N)) = (0)$. Hence using Theorem 3.6, we have the following.

Theorem 3.8 — An *f-n*-ring N is J_2-semisimple if and only if N is a subdirect sum of f.o. near-rings each of which has a left identity and has no N-*l*-subgroups.

If P is an *l*-primitive *l*-ideal of type zero then $P = (I : N)$ for some modular right *l*-ideal I of type zero. For any *l*-ideal J if $J \not\subseteq P$ then $J + P = N = I + J$ and $I \cap J$ is a modular right *l*-ideal of J. Extend $I \cap J$ to a maximal modular right *l*-ideal **K** of I. Since $N/P = (J + P)/P = J/J \cap P$, $J \cap P$ is an l_p-ideal of J and so $J \cap P = (K : I)$. Thus either $J \subseteq P$ or $J \cap P$ is an *l*-primitive *l*-ideal of J of type zero. Hence $J_0(J) \subseteq J \cap J_0(N)$. A similar argument shows that $J_2(J) \subseteq J \cap J_2(N)$. This proves the following theorem.

Theorem 3.9 — Any *l*-ideal I of a J_0-semisimple (J_2-semisimple) *f-n*-ring is again J_0-semisimple (J_2-semisimple).

We now give an example of an *f-n*-ring N in which $J_2(N) \neq J_0(N)$.

Example 2 — Let G_1 and G_2 be two *l*-simple ordered groups. Put $N = G_1 \times G_2$. Then $(N, +)$ is an f.o. group under the ordering $(a, b) \geqslant 0$ if $a > 0$ or $a = 0, b \geqslant 0$. Fix an element $(a, 0)$ in N with $a > 0$. Define multiplication by

$$(c, d) \cdot (e, f) = \begin{cases} (0, 0) \text{ if } (c, d) \leqslant (a, 0) \\ (e, f) \text{ if } (c, d) > (a, 0) \end{cases}$$

Then N is a f.o. near-ring with left identity. $H = (0) \times G_2$ is an N-*l*-subgroup of N with $H^2 = (0, 0)$, $I = ((0, 0))$ is the unique maximal modular *l*-ideal of N, and $(I : N) = (0, 0)$. Thus $J_0(N) = (0, 0)$ where as $J_2(N) = N$.

4. Descending Chain Conditions and J_2-Radical

As in ring theory the descending chain condition (d.c.c.) on *l*-ideals yields the following theorem.

Theorem 4.1 — Let N be an f-n-ring satisfying d.c.c. on l-ideals. Then $J_2(N) = 0$ if and only if N is isomorphic to a finite direct sum of ordered near-rings $\{K_i\}_{i=1}^{n}$, where each K_i has a left identity and has no proper N-l-subgroups.

Even under the d.c.c. on N-l-subgroups, $J_2(N)$ need not be nilpotent as can be seen from the case when N is the ordered ring of even integers. But under some additional conditions the d.c.c. on N-l-subgroups implies the nilpotency of $J_2(N)$.

Lemma 4.2 — Let N be an f.o. near-ring with a minimal N-l-subgroup I with $I^2 \neq (0)$. Then N has no proper N-l-subgroups.

PROOF : $I^2 \neq (0)$ implies that there exists an $a > 0$ in I such that $aI \neq 0$. Consider $H = \{x \in N : |x| \leqslant ay$ for some $y \in I\}$. Then it is easy to see that H is an N-l-subgroup of N and $H \subseteq I$. So $H = I$. Hence there exists $c \in I$ such that $0 < a \leqslant ac$. The set $L = \{x \in N \mid ax = 0\}$ is a right l-ideal of N. If $L \neq (0)$, $I \subseteq L$ and consequently $aI = (0)$, which is not the case. Thus $ax = 0$ implies $x = 0$. Now for any $x > 0$ in N, $ax \leqslant acx$. If $cx < x$ then $acx \leqslant ax$ which implies $a(cx - x) = 0$ and so $cx = x$ is in I. On the other hand if $x \leqslant cx$ then x is in I, as I is convex. That is $N = I$. Hence N hat no proper N-l-subgroups.

Theorem 4.3 — Let N be an ordered near-ring with the d.c.c. on N-l-subgroups, with a left identity and without a nilpotent N-l-subgroup. Then N is an l-primitive near-ring of type two and $J_2(N) = 0$.

PROOF : Let N be an ordered near-ring with the d.c.c. on N-l-subgroups. Then N has a minimal N-l-subgroup I. The fact that I is not nilpotent implies by Lemma 4.2 that N has no proper N-l-subgroups. Thus (0) is the unique modular right l-ideal of type two.

Definition 4.4 — An l-ideal P of an f-n-ring N is said to be strongly prime if for any N-l-subgroup I, $I^2 \subseteq P$ implies $I \subseteq P$.

Theorem 4.5 — Let N be an f-n-ring with left identity and satisfy the d.c.c. on N-l-subgroups. If every prime l-ideal of N is strongly prime then $J_2(N)$ is nilpotent.

PROOF : Let P be a prime l-ideal of N, then $\bar{N} = N/P$ is an ordered near-ring with d.c.c. on N-l-subgroups. Since P is strongly prime $\bar{N}$ has no nilpotent N-l-subgroups. So by Theorem 4.3 $\bar{N}$ is l-primitive of type two and hence P is a modular right l-ideal of N of type two. Thus $J_2(N) \subseteq P$, for every prime l-ideal P. Using same method as in ring theory, it can be shown that under d.c.c. on N-l-subgroups $J_2(N) =$ intersection of all prime l-ideals and hence $J_2(N)$ is nilpotent.

REFERENCES

Blackett, D. W. (1953). Simple and semisimple near-rings. *Proc. Am. math. Soc.*, **4**, 772-85.

Conrad, P. (1965). The lattice of all convex *l*-subgroups of a lattice ordered group. *Czech. Math. J.*, **15**, 101-32.

Fuchs, L. (1963). Partially Ordered Algebraic Systems. Pergamon Press, London.

Johnson, D. G. (1960). A structure theory of a class of l.o. rings. *Acta Math.*, **104**, 163-215.

Pilz, G. F. (1971). Direct sums of ordered near-rings. *J. Algeb.*, **18**, 340-42.

Pierce, R. S. (1956). Radicals in function rings. *Duke Math. J.*, **23**, 253-61.

Steinberg, S. A. (1972). Finitely valued *f*-modules. *Pacific J. Math.*, **40**, 723-37.

Radhakrishna, A. and Bhandari, M. C. (1977). On lattice ordered near-rings. *Pure appl. Math. Sci. (India)*, (to appear).