LONGITUDINAL DISPERSION IN AN ISOTROPIC POROUS MEDIA FLOW

by MOHD A. A. ANSARI, *Department of Mathematics, Banaras Hindu University, Varanasi* 221005

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The possibilities of generalizing the dispersion equations of flow through an isotropic porous media are investigated. Investigations are based on the hypothesis (Bear's hypothesis): "Only that part of each velocity component is of significance, which is either parallel or normal to the mean flow direction". In this paper we have considered and discussed only that part which is parallel to the mean flow direction, i.e. longitudinal dispersion, and presented a generalized closed form solution for dispersion. The analysis is applicable, when the porous media flow is in an isotropic steady state with radioactive decay, neglecting the adsorption. The different types of variations in concentration have been discussed graphically.

1. INTRODUCTION

The mixing of miscible fluids as they follow porous media is generally referred to as dispersion. The coefficient of longitudinal dispersion appearing in the dispersion equation has been analyzed by various investigators. It depends on the pattern, e.g. the velocities and some basic medium characteristics. Many investigators have considered the sum $\alpha' = (\alpha + \alpha_d^*)$ as the coefficient of dispersion which depends on the velocity, the molecular diffusion and the medium characteristics. Obviously matters dealing with essentially one dimensional flow cannot recognize the tensorial nature of α' or of α [when longitudinal molecular diffusion is negligible with respect to the longitudinal dispersion expressed by Taylor (1953, 1954)].

Taylor (1953) in his one dimensional analysis obtained α proportional to $\bar{v}^2$, but Bear and Todd (1960), in their one dimensional analysis suggested $\alpha = a_1 \bar{v}$, i.e. α is proportional to $\bar{v}$ with a_1 being some characteristic medium length, where $\bar{v}$ is velocity.

For steady uniform flow parallel to the x-axis in porous media, the equation for concentration of mass of the displacing substance may be written as :

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} = \alpha \frac{\partial^2 \theta}{\partial x^2} - \lambda \theta \qquad \ldots(1)$$

where

θ = concentration of dispersion

u = average velocity of the fluid

α = coefficient of longitudinal dispersion

x = coordinate parallel to the flow

λ = the decay constant is proportional to the reciprocal of the half life of the radioactive pollutant

t = time.

In this paper we describe mathematical solutions to simplified dispersion problems involving a variable. Harleman and Rumer (1963) have also studied the problem of longitudinal dispersion in an isotropic porous media and developed practical values, neglecting the radioactive decay. In the present paper we have undertaken the study of the same problem considering the radioactive decay, but in particular case, obtained the same equation as by Harleman and Rumer (1963). This case has been examined experimentally by various authors but the same has not been analyzed theoretically. In the present paper it has been discussed theoretically as well as graphically.

It has been concluded that the concentration has the same behaviour at small values of x as well as large values of x for $t > 1$, and it is almost constant between $0 < x < 0.05$. We also observed that the concentration increases as time increases for $x > 0$, and it is constant at $x = 0$ for $t > 1$. It has been also noted that the concentration is half of the constant concentration θ_0, when decay constant is zero. The results obtained have been tabulated and presented graphically to show some variations. Some interesting results have been discussed.

2. Theory

The theory that follows is limited to unsteady longitudinal dispersion in idealized steady one dimensional seepage flows through semi-infinite isotropic porous media. The solutions to be developed predict the concentration of contaminants as a function of time and space, if seepage flow, the longitudinal dispersion coefficient and the source concentration are prescribed, and adsorption is neglected.

Now consider a steady uniform flow through a column packed with a porous media. At $t = 0$, a constant concentration, θ_0, of a tracer substance is introduced into the flow such that at $x = 0$, the concentration is always θ_0 (Fig. 1).

The equation for longitudinal dispersion within semi-infinite column of porous media ($x > 0$) in a unidirectional flow field is given by eqn. (1). The flow in the column is maintained at a constant velocity, u in the x-direction.

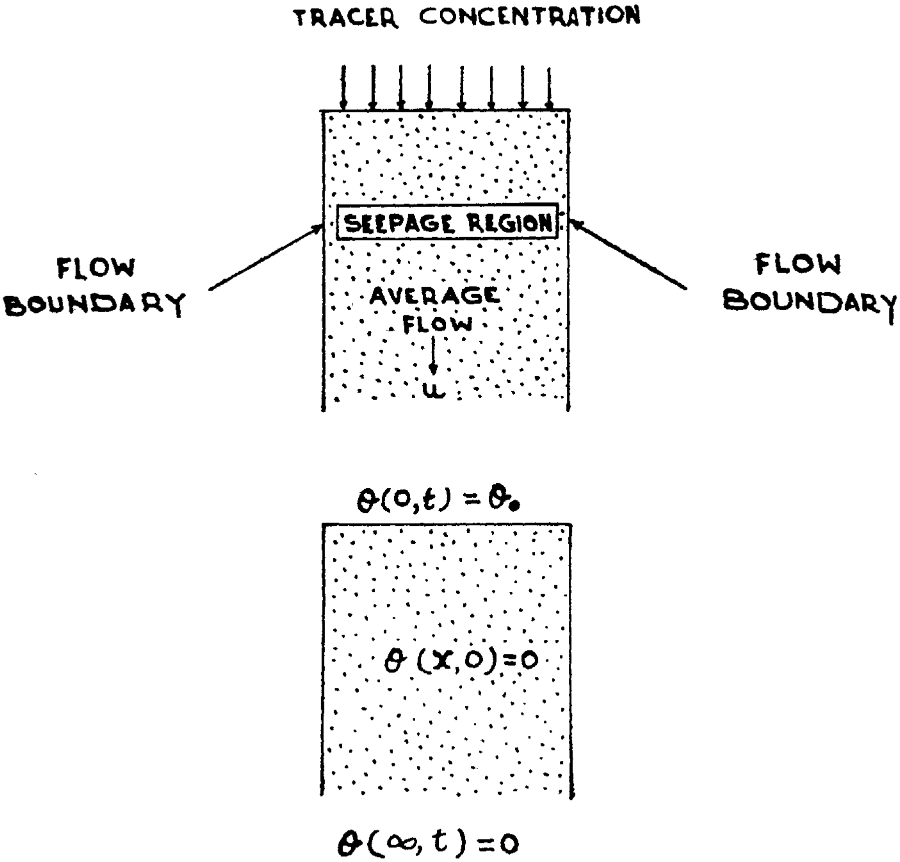

FIG. 1. Semi-infinite column of porous medium ($x > 0$) in unidirectional flow field, the concentration of displacing fluid is θ_0 in the fluid.

We can write the longitudinal dispersion problem as eqn. (1), with initial and boundary conditions :

$$\theta(x, 0) = 0, \qquad t \leqslant 0, \qquad x \geqslant 0 \qquad \text{...(2)}$$

$$\theta(0, t) = \theta_0, \qquad x = 0, \qquad t > 0 \qquad \text{...(3)}$$

$$\theta(\infty, t) = 0, \qquad x = \infty, \qquad t > 0. \qquad \text{...(4)}$$

Let $\theta^*(x, p) = \int_0^\infty e^{-pt}\, dt$ be the transformation with respect to t of the concentration function $\theta(x, t)$ and applying the Laplace transform to eqns.(1) – (4), we have

$$\alpha \frac{\partial^2\theta^*}{\partial x^2} - u\frac{\partial\theta^*}{\partial x} - (\lambda + p)\,\theta^* = 0 \qquad \text{...(5)}$$

$$\theta^*(0, p) = \frac{\theta_0}{p} \qquad \text{...(6)}$$

$$\theta^*(\infty, p) = 0 \qquad \text{...(7)}$$

where p is the transformation parameter. The general solution to eqn. (5) is

$$\theta^* = e^{ux/2\alpha}\left\{c_1 \exp\left(-x\frac{\sqrt{u^2 + 4\alpha(\lambda + p)}}{2\alpha}\right) + c_2 \exp\left(x\frac{\sqrt{u^2 + 4\alpha(\lambda + p)}}{2\alpha}\right)\right\} \qquad \text{...(8)}$$

where c_1 and c_2 are constants of integration.

With the help of eqns. (6) and (7), c_1 and c_2 are determined from eqn. (8) as

$$c_1 = \frac{\theta_0}{p} \qquad \text{...(9)}$$

$$c_2 = 0 \qquad \text{...(10)}$$

and

$$\theta^*(x, p) = \frac{\theta_0}{p}\exp\left[x\left\{\frac{u}{2\alpha} - \left(\frac{u^2}{4\alpha^2} + \frac{\lambda + p}{\alpha}\right)^{1/2}\right\}\right]. \qquad \text{...(11)}$$

By inversion theorem, the solution is found as

$$\theta(x, t) = \theta_0\left(\exp\frac{ux}{2\alpha}\right)\frac{1}{2\pi i}\int_{T-i\infty}^{T+i\infty}\frac{\exp(yt)}{y} \times \exp\left\{-x\left(\beta^2 + \frac{y}{\alpha}\right)^{1/2}\right\}dy \qquad \text{...(12)}$$

where

$$\beta^2 = \frac{u^2}{4\alpha^2} + \frac{\lambda}{\alpha}. \qquad \text{...(13)}$$

Integrating eqn. (12), we obtain (Grobener and Horfreiter 1949-1950)

$$\theta(x, t) = \tfrac{1}{2}\theta_0 \exp\left(\frac{ux}{2\alpha}\right)\left\{\exp(-x\beta) \times \operatorname{erfc}\frac{x-(\sqrt{u^2 + 4\lambda\alpha})t}{2\sqrt{\alpha t}} + \exp(x\beta) \times \operatorname{erfc}\frac{x+(\sqrt{u^2 + 4\lambda\alpha})t}{2\sqrt{\alpha t}}\right\}. \qquad \text{...(14)}$$

Hence we have

$$\theta(x, t) = \tfrac{1}{2}\theta_0 \exp\left[\frac{ux}{2\alpha}\left\{1-\left(1+\frac{4\lambda\alpha}{u^2}\right)^{1/2}\right\}\right]$$

$$\times \operatorname{erfc}\left\{\frac{x - u\left(1+\frac{4\lambda\alpha}{u^2}\right)^{1/2} t}{2\sqrt{\alpha t}}\right\}. \quad \ldots(15)$$

As $t \to \infty$, eqn. (15) becomes

$$\theta(x, t) = \tfrac{1}{2}\theta_0 \exp\left[\frac{ux}{2\alpha}\left\{1-\left(1+\frac{4\lambda\alpha}{u^2}\right)^{1/2}\right\}\right] \quad \ldots(16)$$

This is the steady state solution of the dispersion equation.

Particular case (when $\lambda = 0$)

Equation (13) becomes

$$\theta(x, t) = \tfrac{1}{2}\theta_0 \left\{\operatorname{erfc}\left(\frac{x-ut}{2\sqrt{\alpha t}}\right) + \exp\left(\frac{ux}{\alpha}\right)\right.$$

$$\left.\times \operatorname{erfc}\left(\frac{x+ut}{2\sqrt{\alpha t}}\right)\right\}. \quad \ldots(17)$$

The above equation has been derived by Ogata and Bank (1961) where the displacing fluid has a constant concentration θ_0. They have also shown that the second term of eqn. (17) can be neglected with an error of less than 5%, when $\frac{ux}{\alpha}$ is less than 0.0075.

TABLE I

The values of concentration from eqn. (15) *for* $u = 0.13$ *cm/sec* $\alpha = 0.008$ *cm²/sec and* $\lambda = 0.004$/*sec*

$x \diagdown t$	1	1.5	2
0.00	0.29	0.29	0.29
0.01	0.56	0.57	0.59
0.02	0.53	0.57	0.61
0.03	0.52	0.56	0.59
0.04	0.53	0.56	0.59
0.05	0.57	0.56	0.59
0.06	0.62	0.62	0.68

Then eqn. (17) becomes

$$\theta(x, t) = \tfrac{1}{2}\theta_0 \operatorname{erfc} \frac{x - ut}{2\sqrt{\alpha t}}. \qquad \ldots(18)$$

It is interesting to observe from eqn. (16) that

$$\theta(x, t) = \tfrac{1}{2}\theta_0 \qquad \ldots(19)$$

i.e. the concentration is half of the constant concentration θ_0.

3. Results

The values of concentrations calculated from eqns. (15), (16) and (18) for values of u, α and λ are tabulated in Tables I–IV.

TABLE II

The values of concentration from eqn. (15) *for* $u = 0.13$ *cm/sec*, $\alpha = 0.008$ *cm²/sec, and* $\lambda = 0.004$/*sec for* $t = 0.5$ *sec*

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
θ/θ_0	0.28	0.51	0.46	0.49	0.13	0.14	0.15	0.16	0.18

TABLE III

The values of concentration from eqn. (16) *for* $u = 0.13$ *cm/sec*, $\alpha = 0.008$ *cm²/sec and* $\lambda = 0.004$/*sec*

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
θ/θ_0	0.5	0.53	0.58	0.62	0.67	0.72	0.78	0.82	0.91

TABLE IV

The values of concentration from eqn. (18) *for* $u = 0.13$ *cm/sec* $\alpha = 0.008$ *cm²/sec and* $\lambda = 0$

x \ t	0.5	1	1.5	2
0.00	1.07	1.22	1.30	1.36
0.01	1.02	1.18	1.27	1.34
0.02	0.88	1.03	1.12	1.33
0.03	0.73	1.07	1.18	1.30
0.04	0.65	1.01	1.18	1.26
0.05	0.65	0.95	1.13	1.12
0.06	0.57	0.95	1.13	1.18
0.07	0.07	0.88	1.07	1.18

Solutions of eqns. (15), (16) and (18) are presented graphically in Figs. 2–6 for values of $\alpha = 0.008$ cm²/sec, $u = 0.13$ cm/sec and $\lambda = 0.004$/sec, representatives of those reported in the literature (Harleman *et al.* 1963, Shen 1976). If we put $\lambda = 0$ in eqn. (15) we see that eqn. (14) reduces to eqn. (18).

4. Conclusions and Discussions

From Tables I and II it is evidently clear that the concentrations are not symmetrical in increasing or decreasing order as x increases but the concentration is in increasing order as t increases when $x \geqslant 0.01$. But in Table II, the concentration decreases as x increases for $t < 1$, which is not uniform. In Table I, the concentration is constant at $x = 0$ when $t > 1$. From Table III, in steady state, the

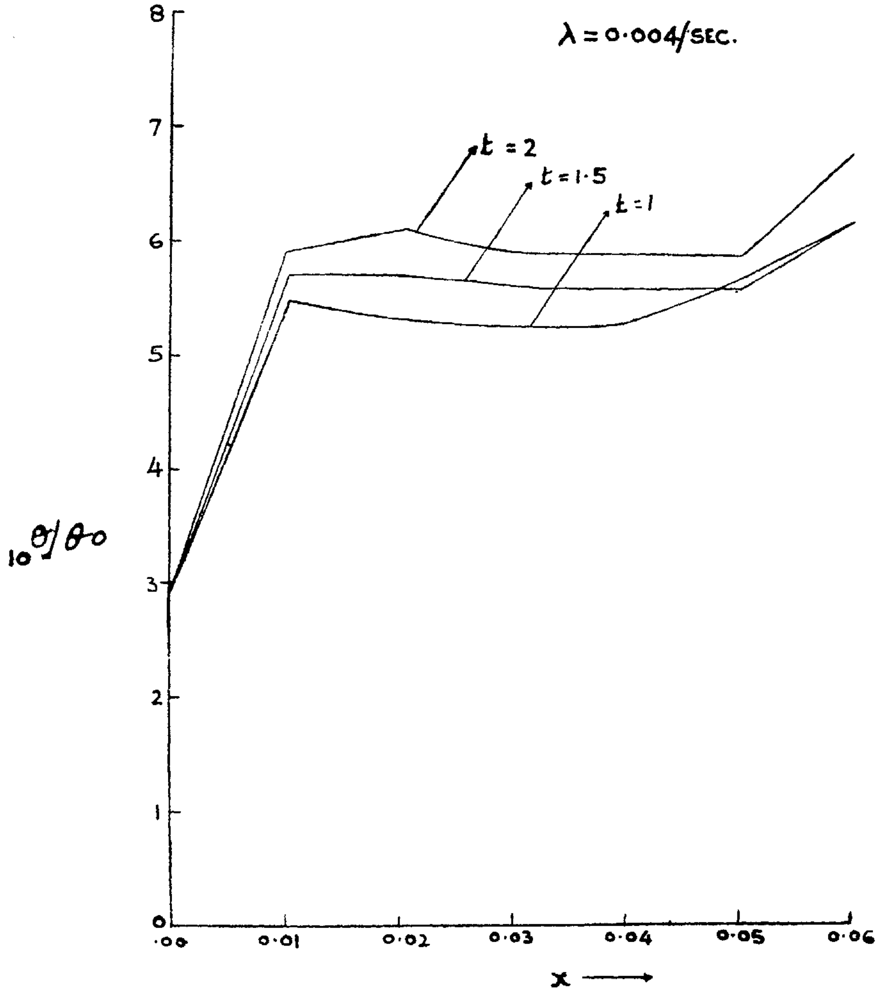

Fig. 2. The variation of θ/θ_0 against the distance along the flow field for various values of time t.

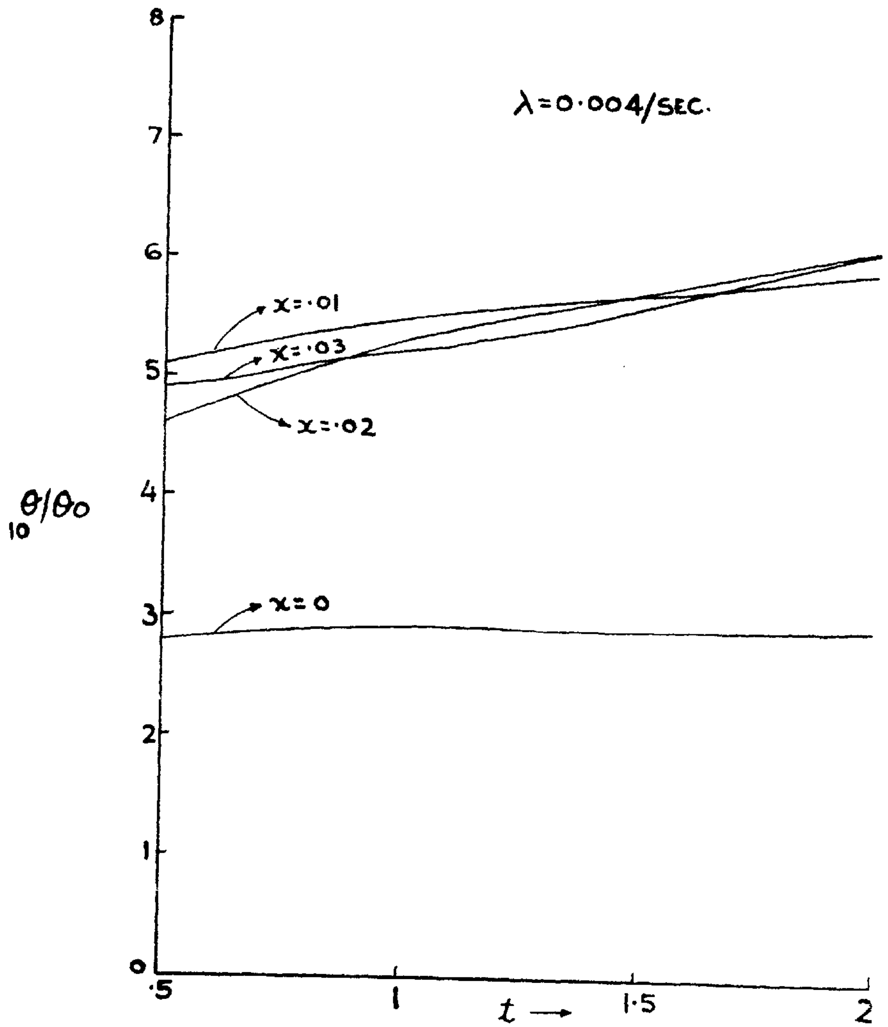

FIG. 3. The variation of θ/θ_0 against the time.

concentration increases as the distance increases along the direction of flow field. From Table IV we have the similar conclusion as in Table I, but the concentration is not constant at $x = 0$ when $t > 1$.

One more interesting result developed by the author at $\lambda = 0$ is that the concentration is half of the constant concentration θ_0.

Proceeding to the analysis of the individual graphs, we conclude as follows :

1. Figs. 2 and 3 show variation of the concentration against the distance and time respectively for a given value of λ (= 0.004/sec). In Fig. 2, we find that the

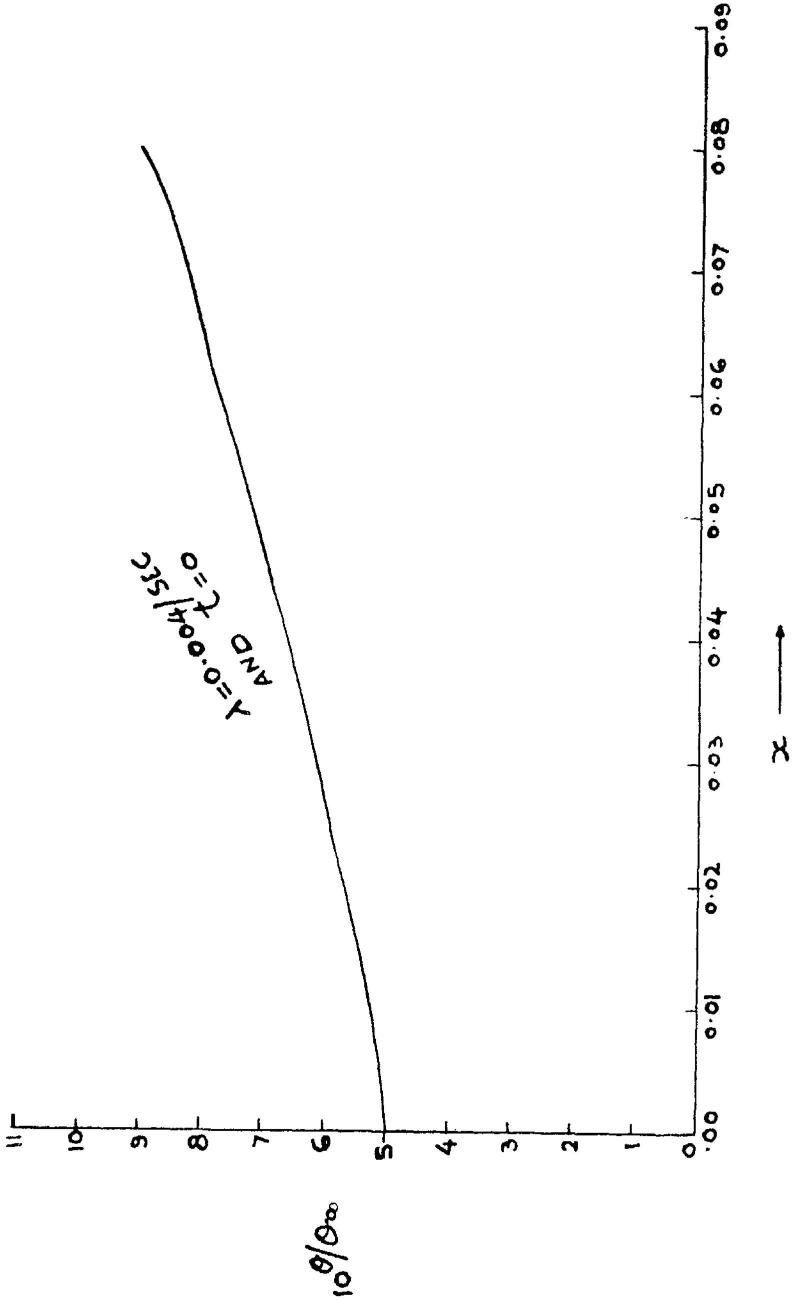

FIG. 4. The variation of the concentration against the distance along the flow field in steady state.

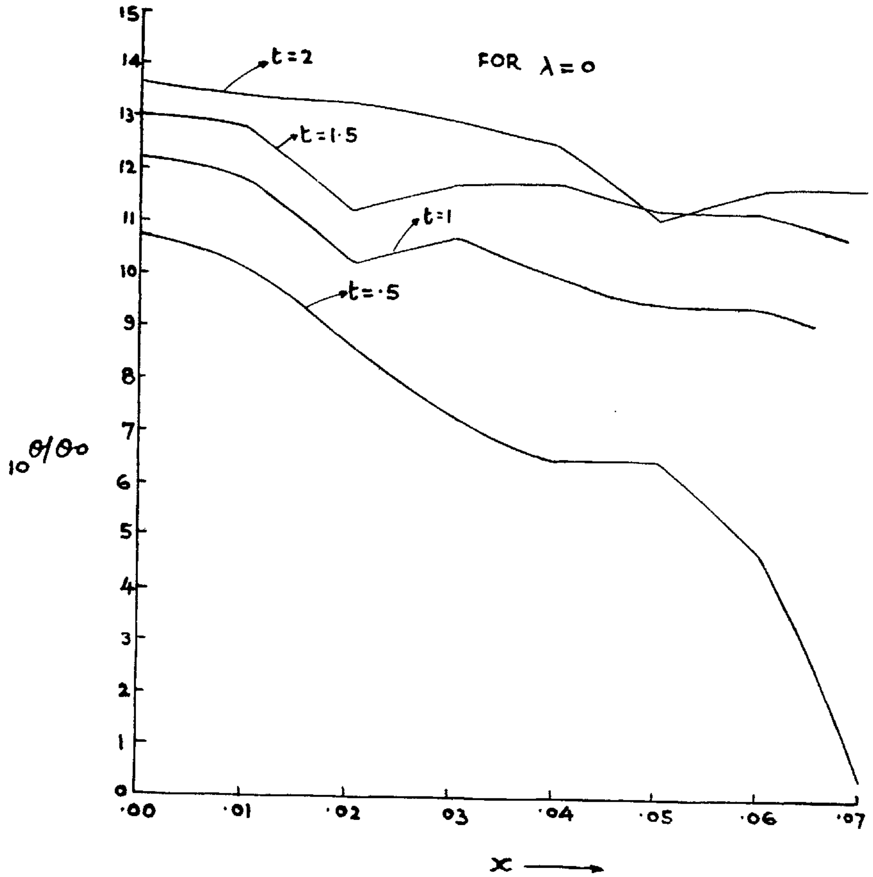

FIG. 5. The variation of θ/θ_0 against the distance along the flow field.

concentration increases between $0 < x < 0.01$ and $0.05 < x < 0.06$ for different values of t, say $t > 1$ and it is almost constant between $0.01 < x < 0.05$ for $t > 1$. But at the intersection point, say 0.048, for $t = 1$ and $t = 1.5$, the concentrations have the same value. In Fig. 3, we find that the concentration increases as t increases for $x \geqslant 0.01$, but at $x = 0$, it is almost constant as t increases.

2. Fig. 4 shows the variation of the concentration against the distance in steady state for a given value of λ (= 0.004/sec). Here it is clear that the concentration increases uniformly as x increases.

3. Figs. 5 and 6 also show the variation of the concentration against the distance and time respectively. In Fig. 5 we find that the concentrations are almost in decreasing order as x increases. For small value of t, say $t = 0.5$ the concentration is in totally decreasing order. The concentrations intersect at 0.0485 cm and 0.052 for

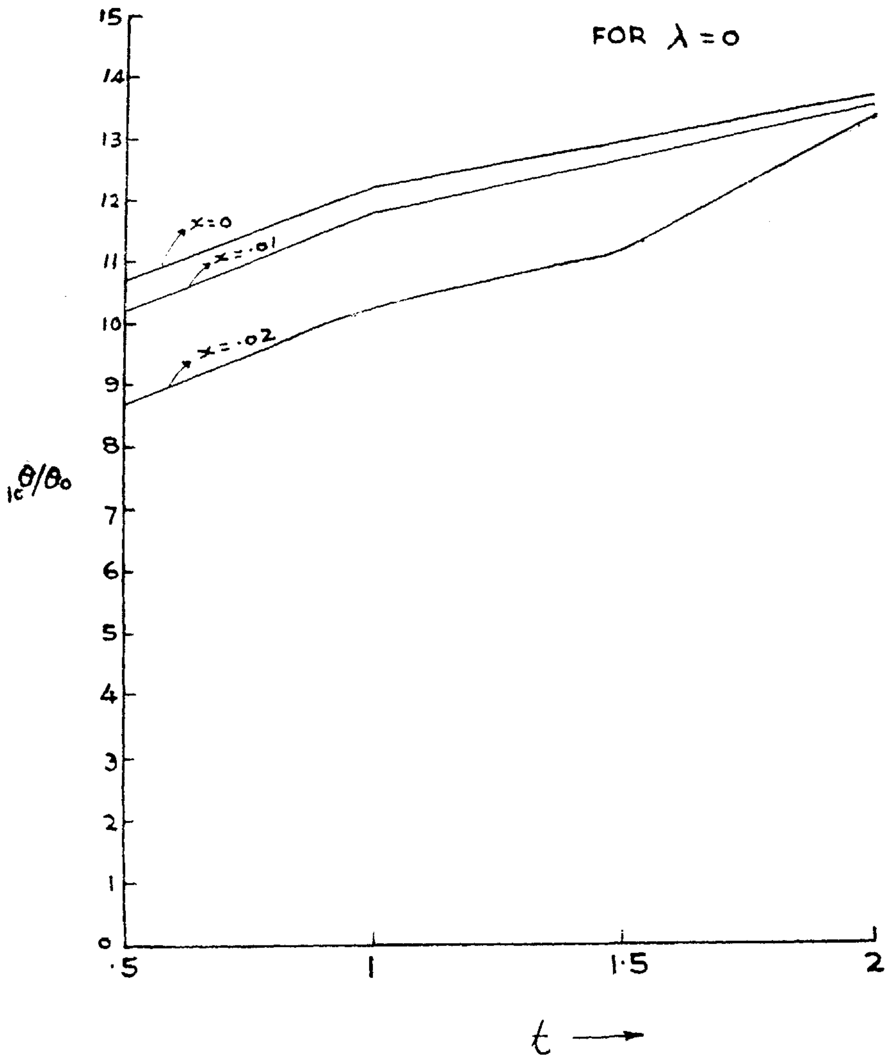

FIG. 6. The variation of θ/θ_0 against the time.

$t = 1.5$ sec and at $t = 2$ sec, have the same value. In Fig. 6 we find that the concentrations increase as t increases.

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