A NEW LOOK AT THE PERMUTATIONS OF THE FIRST *n* NATURAL NUMBERS

by HANSRAJ GUPTA, F.N.A., 402 *Mumfordganj, Allahabad* 211002

(*Received* 24 *April* 1978)

A new way of ordering permutations of the first *n* natural numbers leads to several simplifications in the treatment of the subject. Graphical representations of the inverse and the conjugate of a permutation are offered and several enumeration problems are dealt with. Generating functions for permutations with a given index and degree are obtained. Finally a method is given for finding the number of such permutations without the use of a partition table and the Alter-Curtz-Wang conjecture is partially proved.

The paper has undergone many changes after the original version was read by Professor D. H. Lehmer at the San Diego Conference in December 1976.

1. NOTATION

We write S_n for the set of the first n natural numbers; $(n; r)$ for $\binom{n}{r}$; $p(n, r)$ for the number of partitions of n into r non-negative integers; $p(n)$ for the number of unrestricted partitions of n.

The parts in a partition are invariably written in the non-descending order from left to right.

Results in partition theory are freely used and so are the Tables of Partitions (Gupta *et al.* 1958).

2 ORDERING OF PERMUTATIONS

From any permutation

$$a_1 a_2 a_3 \ldots a_{n-1} \qquad \ldots(2.1)$$

of S_{n-1}, one can derive n permutations of S_n by first writing n on the immediate right of a_{n-1} and then letting it step over

$$a_{n-1}, a_{n-2}, \ldots, a_1$$

in turn. Thus the seven permutations of S_7 that can be derived from the permutation 352146 of S_6 are

3521467
3521476
3521746
3527146
3572146
3752146
7352146

Starting from the unique permutation

1

of S_1, we get

12
21

the two permutations of S_2. Taking each of these 'in turn', we get

123
132
312

213
231
321

the six permutations of S_3. Proceeding in this manner, taking each permutation 'in turn', we can derive from the $(n-1)!$ permutations of S_{n-1} all the $n!$ permutations of S_n. It is noteworthy that the order in which the permutations of S_n thus appear is, for $n > 2$, distinct from the lexicographic order in which the permutations are usually arranged.

3. Summits of a Permutation and their Indices

If in the permutation

$$a_1 a_2 a_3 \ldots a_n \qquad \ldots(3.1)$$

of S_n, for some $k \leqslant n-1$,

$$a_k > a_{k+1} \qquad \ldots(3.2)$$

then we speak of a_k as a summit of (3.1) and of k as the index of this summit. If a_k is not a summit, then its index is taken to be zero.

The sum of the indices of the elements of (3.1) is called the index of (3.1). We shall denote the index of (3.1) by i.

If the indices of the summits of (3.1) are $j_1, j_2, \ldots, j_s$; then we say that the index set of (3.1) is

$$J(s) = \{j_1, j_2, \ldots, j_s\} \qquad \ldots(3.3)$$

Here we invariably assume that

$$1 \leqslant j_1 < j_2 < \ldots < j_s \leqslant n - 1. \qquad \ldots(3.4)$$

Evidently, we have

$$j_1 + j_2 + \ldots + j_s = i. \qquad \ldots(3.5)$$

The only permutation of S_n for which the index set is empty, is

$$1\ 2\ 3 \ldots n \qquad \ldots(3.6)$$

and its index is zero. On the other hand, the only permutation of which the index is $(n; 2)$, is

$$n \quad n-1 \quad n-2 \ldots 3\ 2\ 1. \qquad \ldots(3.7)$$

Directly connected with the index set (3.3) of (3.1), is the 'interval set'

$$R(s) = \{r_1, r_2, \ldots, r_s\} \qquad \ldots(3.8)$$

defined by the relations

$$r_k = j_k - j_{k-1},\ 1 \leqslant k \leqslant s; \text{ with } j_0 = 0. \qquad \ldots(3.9)$$

The elements of the interval set are all positive integers not necessarily distinct. The elements of the index set are, however, distinct positive integers.

Relation (3.5) implies that

$$s\, r_1 + (s-1)\, r_2 + \ldots + 2\, r_{s-1} + r_s = i. \qquad \ldots(3.10)$$

4. The Right Hand and Left Hand Degrees of the Elements of a Permutation

From among the a's on the right side of a_k in (3.1), $1 \leqslant k \leqslant n$, let f_k denote the number of those that are $< a_k$. Similarly let g_k denote the number of a's which are $> a_k$ and are to its left in (3.1). Then f_k and g_k are said to be the right hand and left hand degrees respectively of a_k in (3.1). We write

$$F = F[a_1, a_2, \ldots, a_n] = [f_1, f_2, \ldots, f_n] \qquad \ldots(4.1)$$

and

$$G = G[a_1, a_2, \ldots, a_n] = [g_1, g_2, \ldots, g_n] \qquad \ldots(4.2)$$

to mean that the right hand degree $F(a_k)$ of a_k in (3.1) is f_k and its left hand degree $G(a_k)$ in (3.1) is g_k.

We also write

$$B = B[1, 2, 3, \ldots, n] = [b_1, b_2, b_3, \ldots, b_n] \qquad \ldots(4.3)$$

to mean that the right hand degree of k in (3.1) is b_k.

In what follows, we use f_k, g_k, b_k and F, G, B in this sense throughout and call F, G, B the 'degree vectors' of (3.1). Where nothing is specifically mentioned, degree will mean the right hand degree of an element.

Theorem 1 — For each $k \leqslant n$, we have

$$a_k + g_k = k + f_k.$$

PROOF : The number of a's which are $< a_k$ and appear on its left in (3.1) is $k - 1 - g_k$. Hence

$$f_k + k - 1 - g_k = a_k - 1.$$

From this the theorem follows immediately.

Corollary — For each k,

$$1 + f_k \leqslant a_k \leqslant k + f_k \leqslant n.$$

PROOF : Evidently

$$0 \leqslant f_k \leqslant n - k \quad \text{and} \quad a_k > f_k.$$

Hence the inequalities follow directly.

Theorem 2 — For any permutation of S_n,

$$\sum_{k=1}^{n} f_k = \sum_{k=1}^{n} g_k = \sum_{k=1}^{n} b_k.$$

PROOF : From Theorem 1, we have

$$\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} g_k = \sum_{k=1}^{n} k + \sum_{k=1}^{n} f_k$$

whence the left hand equality follows at once. The right hand equality follows from the fact that the b's are just the f's in some order.

The theorem shows that the sum of the elements of each of the three degree vectors F, G, B is the same. We will denote this common sum by d and call it the 'degree' of the permutation (3.1).

Theorem 3 — a_k is a summit of (3.1), if and only if

$$f_k > f_{k+1}.$$

PROOF : If a_k is a summit, then we assert that

$$f_k \geqslant 1 + f_{k+1}.$$

Since a_k is a summit,

$$a_{k+1} < a_k.$$

Hence, the number of a's which are $< a_k$ and are on its right in (3.1) is certainly at least one more than the number of a's which are $< a_{k+1}$ and appear on the right side of a_{k+1} in (3.1).

On the other hand, if

$$a_k < a_{k+1}$$

then it is clear that

$$f_k \leqslant f_{k+1}.$$

This proves the theorem.

Theorem 3 will be of immense use to us later.

Definition — A permutation will be said to be of the type (s, i, d) if it has s summits, the index i and is of degree d.

Example — The permutation 4367125 of S_7 is of type (2, 5, 11).

5. Determination of a Permutation From Its Degree Vector B

Let b_k be the number of a's which are $< k$ and are on its right side in (3.1). Then the permutation is uniquely determined from its degree vector B, in the following manner.

Mark n positions in a line :

0 0 0 0 ... 0 0 0

Let n move along the line from the right, leave b_n positions empty and occupy the next position. Next, let $(n - 1)$ move along the line from the right like n, leave b_{n-1} 'unoccupied' positions empty and occupy the next 'empty' position. Continue the operation with $(n - 2)$, $(n - 3)$, ..., 3, 2, 1 in turn and you get the desired permutation. The rule is simple: When k has its turn, it moves along the line from the right, leaves b_k 'unoccupied' positions empty and occupies the next 'empty' position.

Example — Let $B = [0, 0, 2, 0, 2, 2, 3, 1]$, then the required permutation is constructed as follows :

0	0	0	0	0	0	0	0
0	0	0	0	0	0	8	0
0	0	0	7	0	0	8	0
0	0	0	7	6	0	8	0
0	0	5	7	6	0	8	0
0	0	5	7	6	0	8	4
3	0	5	7	6	0	8	4
3	0	5	7	6	2	8	4
3	1	5	7	6	2	8	4

The desired permutation appears at the last step.

We have a nice criterion for determining whether or not a given figure will be a summit. It can be stated in the form :

If the given figure has an empty position on its immediate right when it appears for the first time in the above scheme, then it is a summit, otherwise not.

In our example, 8, 7, 6 and 3 alone are summits.

It will be observed that for the above construction to work, it is necessary and sufficient that for each $k \leqslant n$, we have

$$0 \leqslant b_k \leqslant k - 1. \qquad \ldots(5.1)$$

6. The Determination of a Permutation from its F Vector

The method is very similar to the one in the preceding section. We first note that an F vector is legitimate if and only if

$$0 \leqslant f_k \leqslant n - k.$$

The following example describes the scheme in this case.

Let $F = [2, 3, 0, 1, 2, 0, 0]$.

Evidently $n = 7$. We mark 7 positions and number them from left to right. Now a_1 moves along the line from left to right, leaves f_1 positions empty and occupies the next position. Next a_2 moves along the line from left to right like a_1, leaves f_2 'unoccupied' positions empty and occupies the next 'empty' position. The operation is continued with the remaining a's. The value of any a is given by the number of the position occupied by it.

In our example, we have

1	2	3	4	5	6	7
0	0	0	0	0	0	0
0	0	a_1	0	0	0	0
0	0		0	a_2	0	0
a_3	0		0		0	0
	0		a_4		0	0
	0				0	a_5
	a_6				0	
					a_7	

The required permutation is thus seen to be 3 5 1 4 7 2 6. For any legitimate F vector the permutation is again uniquely determined.

7. The Position of a Given Permutation in our Ordering

Given a permutation of S_n, the following algorithm determines the number of permutations which precede it from among the $n!$ permutations of S_n.

Elements :	1	2	3	4	...	n
b's	b_1	b_2	b_3	b_4	...	b_n
	m_1	m_2	m_3	m_4	...	m_n

We start with $m_1 = b_1 = 0$, and then calculate the values of the other m's from the formula :

$$m_k = k m_{k-1} + b_k, \quad k = 2, 3, \ldots, n. \qquad \ldots(7.1)$$

In the algorithm, m_n gives the number of permutations which precede the given permutation (that is the one corresponding to the B vector) in our ordering.

From (7.1), it will be readily seen that

$$m_n = b_n + nb_{n-1} + n(n-1)\, b_{n-2} + \ldots + n!\, b_1$$

$$= n! \sum_{k=1}^{n} \frac{b_k}{k!}. \qquad \ldots(7.2)$$

The justification for the algorithm is provided by the fact that the given permutation arises from a permutation of S_{n-1} obtainable from the given permutation by the deletion of n. If this permutation had m_{n-1} permutations preceding it in its own set, then the given permutation is preceded by $nm_{n-1} + b_n$ permutations in the derived set.

Example — Let us determine the position of the permutation

3521746

among the permutations of S_7.

The algorithm takes the form:

	1	2	3	4	5	6	7
b's	0	1	2	0	3	0	2
m's	0	1	5	20	103	618	4328

The given permutation is, therefore, the 4329th in the set of 7! permutations of S_7.

Given the position of a permutation, the reverse process will give its B vector, from which the permutation itself can be determined. From the above algorithm, it will also be clear that if the permutations of S_n are arranged in our way, then their B vectors are in lexicographic order but not so their F vectors.

Thus for $n = 4$, we have

Position	Permutation	B vector	F vector
1	1234	0000	0000
2	1243	0001	0010
3	1423	0002	0200
4	4123	0003	3000
5	1324	0010	0100
6	1342	0011	0110
7	1432	0012	0210
8	4132	0013	3010
9	3124	0020	2000
10	3142	0021	2010
11	3412	0022	2200
12	4312	0023	3200
13	2134	0100	1000
14	2143	0101	1010
15	2413	0102	1200
16	4213	0103	3100
17	2314	0110	1100
18	2341	0111	1110
19	2431	0112	1210
20	4231	0113	3110
21	3214	0120	2100
22	3241	0121	2110
23	3421	0122	2210
24	4321	0123	3210

8. The Graphs of a Permutation, Its Conjugate and of Its Inverse

The graph of the permutation in (3.1) is obtained by plotting in the usual way on a graph paper, the points

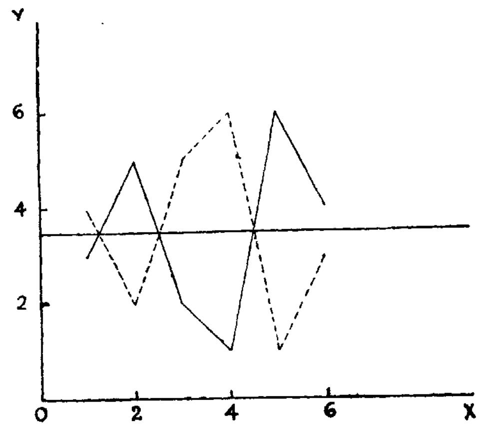
$$(1, a_1), (2, a_2), (3, a_3), \ldots, (n, a_n) \qquad \ldots(8.1)$$

and joining them by straight lines in the order in which the 'abscissae increase'.

The images of the points (8.1), in the lines

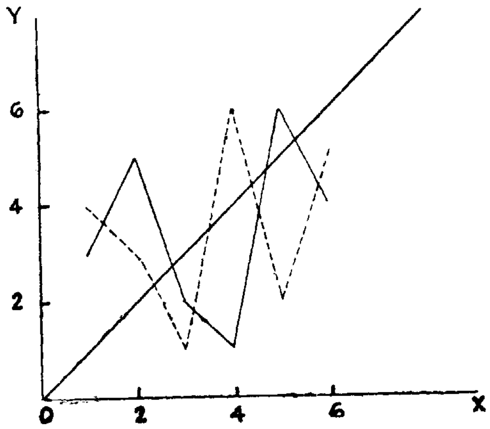
$$2y = n + 1 \text{ and } y = x$$

provide respectively the graphs of the 'conjugate' and the 'inverse' of the permutation in (3.1). In Figs. 1 and 2 are shown, for example, the graphs of the permutation 352164 and those of its conjugate and inverse.



Graph of 352164 ——
Graph of the conjugate - - - -

FIG. 1.



Graph of 352164 ——
Graph of the inverse - - - -

FIG. 2.

9. Properties of the Conjugate of a Permutation

Let

$$h_1h_2h_3 \ldots h_n \qquad \ldots(9.1)$$

be the conjugate of (3.1). Denote by $I(a_k)$ the index and by $F(a_k)$ the degree of a_k in (3.1). Similarly, let $I(h_k)$ and $F(h_k)$ denote respectively the index and the degree of h_k in (9.1). Then, we have

Theorem 4 — (i) $h_k + a_k = n + 1$;

(ii) $I(h_k) + I(a_k) = k$, if $1 \leqslant k \leqslant n - 1$,

$= 0$, for $k = n$;

(iii) $F(h_k) + F(a_k) = n - k$, for each $k \leqslant n$.

PROOF : (i) is a direct consequence of the definition. The definition also implies that for any two distinct positive integers k_1 and k_2 each $\leqslant n$,

$$a_{k_1} > a_{k_2} \text{ or } < a_{k_2} \qquad \ldots(9.2)$$

according as $h_{k_1} < h_{k_2}$ or $> h_{k_2}$.

(ii) and (iii) follow readily from (9.2).

Furthermore, we have

$$\sum_{k=1}^{n} I(h_k) = \sum_{k=1}^{n} k - \sum_{k=1}^{n} I(a_k);$$

and

$$\sum_{k=1}^{n} F(h_k) = \sum_{k=1}^{n-1} (n-k) - \sum_{k=1}^{n} F(a_k).$$

Therefore, if (3.1) is of the type (s, i, d), then (9.1) is of the type

$$(n-1-s, (n;\, 2) - i, (n;\, 2) - d). \qquad \ldots(9.3)$$

Example — The permutation 4367125 is of the type (2, 5, 11). Its conjugate 4521763 is of the type (4, 16, 10).

10. THE TYPE OF A DERIVED PERMUTATION

Let the permutation

$$a_1 a_2 a_3 \ldots a_{n-1} \qquad \ldots(10.1)$$

of S_{n-1} be of the type (s, i, d).

First consider the permutation derived from (10.1) by writing n to the immediate right of a_{n-1}. Since in this operation no new summits are introduced nor is the index or the degree of any element altered, while the index and the degree of n is each zero, the derived permutation is still of the type (s, i, d).

Let us next insert n between a_{n-k-1} and a_{n-k} and assume that there are exactly t summits of (10.1) to its right, $k \geqslant 1$, $t \geqslant 0$. Then, two cases arise :

(1) *When* (10.1) *has a summit at* a_{n-k-1}

In this case, the derived permutation has still s summits, its index i^* is given by

$$i^* = i + t + 1 \qquad \ldots(10.2)$$

and its degree d^* is given by

$$d^* = d + k. \qquad \ldots(10.3)$$

This is because with the insertion of n, the summit has moved from a_{n-k-1} to n and the index of each of the t summits to its right has increased by 1. Moreover, the degree of n is k while the degrees of all other elements have remained unaltered.

The derived permutation is, therefore, of the type $(s, i + t + 1, d + k)$.

(2) *When a_{n-k-1} is not a summit of* (10.1)

The derived permutation has now a new summit at n with index $(n - k)$ and reasoning as before, we have

$$i^* = i + t + (n - k) \qquad \text{...(10.4)}$$

$$d^* = d + k. \qquad \text{...(10.5)}$$

The derived permutation is clearly of the type $(s + 1, i + t + n - k, d + k)$.

It will be seen that of the n permutations derivable from (10.1) :

(i) Exactly $(s + 1)$ have s summits each, the remaining $(n - s - 1)$ have $(s + 1)$ summits each; ...(10.6)

(ii) The indices of the derived permutations are all distinct and run from i to $(n + i - 1)$ in some order; ...(10.7)

and (iii) The degrees of the derived permutations are distinct and run from d to

$$(n + d - 1). \qquad \text{...(10.8)}$$

Example — Let us consider the eight permutations that can be derived from the permutation 3521647 of the type (3, 10, 7).

The work of writing the indices and the degrees of the derived permutations can be presented conveniently in the form :

Permutation	with s summits	index	with $(s + 1)$ summits	index	degree
35216478	×	10			7
35216487			×	17	8
35216847	×	11			9
35218647			×	16	10
35281647	×	12			11
35821647	×	13			12
38521647			×	15	13
83521647			×	14	14

It will be observed that the degrees can be entered in the table in a straightforward manner.

For the indices, the derived permutations are divided into two subsets—one consisting of those with s summits, and the other of those with $(s+1)$ summits. In the case of the first subset, the indices increase from top down, in the case of the second from bottom up.

11. Permutations with s Summits

Let $t(n, s)$ denote the number of those permutations of S_n which have exactly s summits. Then, we have

Theorem 5 — For $n \geqslant 2$, and $1 \leqslant s \leqslant n-1$,

$$t(n, s) = (s+1)\, t(n-1, s) + (n-s)\, t(n-1, s-1),$$

with

$$t(m, 0) = 1 \text{ for each } m > 0,$$

$$t(m, k) = 0 \text{ if } 0 < m < k.$$

The theorem is a direct consequence of (10.6).

Theorem 6 — For all nonnegative integers n and s,

$$t(n, s) = \sum_{k=0}^{s} (-1)^k (n+1; k)(s-k+1)^n.$$

PROOF: The expression on the right can be shown to satisfy the recurrence relation in Theorem 5.

Note: Let p be any prime. Then Theorem 6 gives

$$t(p-1, s) \equiv 1 \pmod{p}.$$

Since

$$\sum_{s=0}^{p-2} t(p-1, s) = (p-1)!$$

we get

$$(p-1)! \equiv p-1 \pmod{p}.$$

This is Wilson's theorem.

The numbers $t(n, s)$ are the Eulerian numbers and have been studied in detail by several writers. It will not, however, be out of place here to give a short table of the values of $t(n, s)$.

Values of $t(n, s)$

$s \backslash n$	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1		1	4	11	26	57	120	247	502
2			1	11	66	302	1191	4293	14608
3				1	26	302	2416	15619	88234
4					1	57	1191	15619	156190
5						1	120	4293	88234
6							1	247	14608
7								1	502
8									1

The symmetry in the column entries is explained by (9.3).

12. Permutations with a Given Index

Let $u(n, i)$ denote the number of those permutations of S_n which have the given index i. Then we have

Theorem 7 — For $n \geqslant 1$, and $0 \leqslant i \leqslant (n; 2)$,

$$u(n, i) = \sum_k u(n-1, k), \text{ with } u(0,0) = 1$$

where k runs over nonnegative integers from $(i - n + 1)$ to i. The theorem is a direct consequence of (10.7).

We can also give a generating function for $u(n, i)$.

Let $$U(x, n) = \sum_{i \geqslant 0} u(n, i)\, x^i.$$

Then Theorem 7 implies that

$$\begin{aligned} U(x, n) &= (1 + x + x^2 + \ldots + x^{n-1}) \,.\, U(x, n-1), \\ &= (1 + x)(1 + x + x^2) \ldots (1 + x + x^2 + \ldots + x^{n-1}), \\ &= (1 - x)(1 - x^2)(1 - x^3) \ldots (1 - x^n)/(1 - x)^n. \qquad \ldots(12.1) \end{aligned}$$

13. Permutations with a given Degree

Let $\nu(n, d)$ denote the number of permutations of S_n which are of degree d.

Then (10.8) clearly shows that the number-theoretic functions $u(n, k)$ and $\nu(n, k)$ have the same recurrence relation and the same initial value. They are therefore identical, though they arise independently. We give a short table of values for the two.

$k \setminus n$	1	2	3	4	5	6
0	1	1	1	1	1	1
1		1	2	3	4	5
2			2	5	9	14
3			1	6	15	29
4				5	20	49
5				3	22	71
6				1	20	90
7					15	101
8					9	101
9					4	90

One can use (9.3) to supply the missing entries in the last two columns.

The problem of this section can also be dealt with as follows. Evidently $\nu(n, d)$ is the number of solutions of the Diophantine equation :

$$b_1 + b_2 + \ldots + b_n = d. \qquad \ldots(13.1)$$

where

$$0 \leqslant b_k \leqslant k - 1. \qquad \ldots(13.2)$$

Letting b_n take the values 0, 1, 2, ..., $n-1$ in succession in the equation

$$b_1 + b_2 + \ldots + b_{n-1} = d - b_n \qquad \ldots(13.3)$$

we readily get the recurrence formula for $\nu(n, d)$.

When however d is fixed and n is arbitrary, the following procedure is better suited to our needs.

Since b_1 is necessarily zero, (13.1) provides in a way a decomposition of d into at most $(n-1)$ parts. If the largest of these parts is c and it occurs m times in the decomposition, we select m of the numbers

$$b_{c+1}, b_{c+2}, \ldots, b_n$$

(which are the only ones eligible for the value c) and take each of them equal to c. This requires that we have

$$c \leqslant n - 1 \text{ and } m \leqslant n - c. \qquad \ldots(13.4)$$

The m b's selected go out of our list and we repeat the process with the next largest part in the decomposition and with the b's now left over. This is continued till all the parts in the decomposition have been dealt with. If at any stage conditions such as those in (13.4) are not satisfied, the decomposition makes no contribution to the number $\nu(n, d)$. Since we have to take the largest part at each stage, it will be enough first to list all the partitions of d into at most $(n-1)$ parts and the largest part not exceeding $(n-1)$.

Example — Let us find the number of permutations of S_n which are of degree 5. We first note that for $n < 4$, no permutation of S_n can have the degree 5. We can, therefore assume that $n \geqslant 4$.

The partitions of 5 are

(i) 5; (ii) $1+4$; (iii) $2+3$; (iv) $1+2+2$; (v) $1+1+3$; (vi) $1+1+1+2$; (vii) $1+1+1+1+1$.

(i) makes a contribution only if $n \geqslant 6$.

The eligible b's are

$$b_6, b_7, \ldots, b_n.$$

Only one of these has to be selected. The required contribution is

$$(n-5; 1).$$

(ii) This makes a contribution only if $n \geqslant 5$.

We select one b from the following and take it equal to 4. The b's are

$$b_5, b_6, \ldots, b_n.$$

After the selection $(n-5)$ of these b's will be left and b_2, b_3, b_4 will also be eligible to take the value 1. The required contribution will thus be

$$(n-4; 1)\,(n-2; 1).$$

Proceeding in this manner, the contributions from the other partitions of 5, will be found to be

(iii) $(n-3; 1)\,(n-3; 1)$ for $n \geqslant 4$;

(iv) $(n-2; 2)\,(n-3; 1)$ for $n \geqslant 4$;

(v) $(n-3; 1)\,(n-2; 2)$ for $n \geqslant 4$;

(vi) $(n-2; 1)\,(n-2; 3)$ for $n \geqslant 4$;

(vii) $(n-1; 5)$ for $n \geqslant 6$.

Hence $\nu(4, 5) = 3$, $\nu(5, 5) = 3 + 4 + 6 + 6 + 3 = 22$;

and for $n \geqslant 6$, $\nu(n, 5)$ is the sum of the contributions noted under (i) – (vii) above.

Let
$$d = m_1c_1 + m_2c_2 + \ldots + m_rc_r \qquad \ldots(13.5)$$

be any partition of d with

$$1 \leqslant c_1 < c_2 < \dots < c_r \leqslant n-1,$$

and
$$m_i \leqslant n - c_r,\ m_{r-1} + m_r \leqslant n - c_{r-1}, \dots,$$
$$m_1 + m_2 + \dots + m_r \leqslant n - c_1.$$

Then the contribution which (13.5) makes to $\nu(n, d)$ is given by

$$(n - c_r; m_r)(n - c_{r-1} - m_r; m_{r-1})(n - c_{r-2} - m_{r-1} - m_r; m_{r-2}) \dots$$
$$\dots (n - c_1 - m_2 - m_3 - \dots - m_r; m_1) \qquad \dots(13.6)$$

We leave it to the reader to prove that for $n > d$, $\nu(n, d)$ is a polynomial in n of degree d.

14. Permutations with a given Index and a given Number of Summits

Let i be the given index and s the given number of summits. Evidently then, the required number is the number of solutions of the Diophantine equation

$$s r_1 + (s-1) r_2 + \dots + r_s = i,\ n > r_1 + r_2 + \dots + r_s; \qquad \dots(14.1)$$

in positive integers r.

This is the same as the number of solutions of the Diophantine equation

$$s x_1 + (s-1) x_2 + \dots + x_s = i - (s+1; 2) \qquad \dots(14.2)$$

in nonnegative integers x.

The left side of (14.2) provides a partition of $i - (s+1; 2)$ into parts the largest of which does not exceed s or what is the same thing into at most s parts. Hence the required number is

$$p(i - (s+1; 2), s). \qquad \dots(14.3)$$

Here we take $p(0, s) = 1$ and $p(i - (s+1; 2), s) = 0$ if $i < (s+1; 2)$.

15. Permutations with a given Index Set

Let $P_n(j_1, j_2, \dots, j_s)$ denote the number of permutations of S_n which have the index set

$$J(s) = \{j_1, j_2, \dots, j_s\}$$

where
$$1 \leqslant j_1 < j_2 < \dots < j_s \leqslant n-1.$$

When the index set is empty, we have

$$P_n(\phi) = 1. \qquad \dots(15.1)$$

Theorem 8 — For any $n > j_s$,

$$P_n(j_1, j_2, \ldots, j_s) + P_n(j_1, j_2, \ldots, j_{s-1}) = (n; j_s) \,.\, P_{j_s}(j_1, j_2, \ldots, j_{s-1}).$$

PROOF : From the set of n symbols 1, 2, 3, ..., n select any $(n - j_s)$, arrange them in ascending order and take them to form the section

$$a_{j_s+1}\ a_{j_s+2}\ \ldots\ a_n$$

of some permutation (3.1) of S_n.

Arrange the remaining j_s symbols so that they have summits at

$$a_{j_1}, a_{j_2}, \ldots, a_{j_{s-1}}.$$

If in the resulting permutation

$$a_{j_s} > a_{j_s+1}$$

we get a permutation of S_n with the given index set. Otherwise, we get a permutation of S_n with the index set

$$\{j_1, j_2, \ldots, j_{s-1}\}.$$

Hence

$$P_n(j_1, j_2, \ldots, j_s) + P_n(j_1, j_2, \ldots, j_{s-1}) = (n; n - j_s) \,.\, P_{j_s}(j_1, j_2, \ldots, j_{s-1}).$$

This is the theorem.

In particular, we have

$$P_n(j_1) = (n; j_1) - 1;$$

$$P_n(j_1, j_2) = (n; j_2) \,.\, P_{j_2}(j_1) - P_n(j_1)$$

$$= (n; j_2)\,(j_2; j_1) - (n; j_2) - (n; j_1) + 1.$$

Proceeding in this manner, one can easily obtain an explicit expression for $P_n(j_1, j_2, \ldots, j_s)$ for any s, in terms of combinatory functions. The general result can, however, be obtained directly as follows.

The s summits break up the permutation into $(s + 1)$ sections in each of which the elements are in an ascending order. The last element of each section is, however, strictly greater than the first element of the section on its right (if there is one).

From the set of n symbols 1, 2, 3, ..., n select $(n - j_s)$, arrange them in ascending order and let them form the last section. From the remaining j_s symbols, select $j_s - j_{s-1}$, arrange them in ascending order as before, and let them form the section preceding the last. Continue this process of selection and arranging in order to form a new section. Then the second section from the left is formed from $j_2 - j_1$ symbols selected from the j_2 available at the stage. The remaining j_1 symbols provide the first section from the left.

The procedure described above, will give us all those permutations of S_n, the index sets of which are subsets of $J(s)$, including of course the empty set and the set $J(s)$ itself but no permutation the index set of which is not a subset of $J(s)$.

Hence, the number $N_n(J(s))$ of permutations of S_n the index sets of which are subsets of $J(s)$ is given by

$$\begin{aligned} N_n(J(s)) &= (n; n - j_s)\,(j_s; j_s - j_{s-1}) \ldots (j_2; j_2 - j_1) \\ &= (n; j_s)\,(j_s; j_{s-1}) \ldots (j_2; j_1). \qquad \ldots(15.2) \end{aligned}$$

Given any index set J^*, (15.2) enables us to give an expression for $N_n(J^*)$—the number of those permutations of S_n the index sets of which belong to the class of subsets of J^*. If $| J^* | = k$, the number of such subsets is 2^k.

Letting J^* run through the subsets of $J(s)$, (15.2) gives

$$N_n(J(s)) = \sum_{J^*} P_n(J^*). \qquad \ldots(15.3)$$

Using the inclusion and exclusion principle, we then have

$$P_n(J(s)) = \sum_{J^*} (-1)^{s-|J^*|} N_n(J^*). \qquad \ldots(15.4)$$

This compares favourably with the corresponding formula given by Foulkes (1976). For $s = 3$, (15.4) gives

$$\begin{aligned} P_n(j_1, j_2, j_3) &= N_n(j_1, j_2, j_3) - N_n(j_1, j_2) - N_n(j_1, j_3) - N_n(j_2, j_3) \\ &\quad + N_n(j_1) + N_n(j_2) + N_n(j_3) - 1; \\ &= (n; j_3)\,(j_3; j_2)\,(j_2; j_1) - (n; j_2)\,(j_2; j_1) \\ &\quad -(n; j_3)\,(j_3; j_1) - (n; j_3)\,(j_3; j_2) + (n; j_1) \\ &\quad + (n; j_2) + (n; j_3) - 1. \end{aligned}$$

Some important consequences of Theorem 8

We have

$$P_n(j_1, j_2, \ldots, j_s) = (n; j_s)\, P_{j_s}(j_1, \ldots, j_{s-1}) - P_n(j_1, \ldots, j_{s-1}). \qquad \ldots(15.5)$$

Evidently the expression on the right in (15.5) vanishes if n is replaced therein by j_s. Similarly, $P_n(j_1, \ldots, j_{s-1})$ vanishes for $n = j_{s-1}$.

This in turn implies that the expression on the right in (15.5) vanishes also at $n = j_{s-1}$. The same reasoning will show that

$$P_n(j_1, j_2, \ldots, j_s) \text{ vanishes for } n = j_k,\ k = 1, 2, \ldots, s. \qquad \ldots(15.6)$$

Moreover (15.5) implies that we can write

$$P_n(j_1, j_2, \ldots, j_s) = q_s(n; j_s) + \ldots + (-1)^k q_{s-k}(n; j_{s-k}) + \ldots + (-1)^{s-1} q_1(n; j_1) + (-1)^s \qquad \ldots(15.7)$$

where the q's are independent of n.

It is thus clear that

$P_n(j_1, j_2, \ldots, j_s)$ is a polynomial of degree j_s in n and it assumes the value zero for each of the values $n = j_1, \ldots, j_s$. This simple observation enables us to evaluate the q's when the j's are given. For example, we have

$$P_n(2, 4, 6, 8) = q_4(n; 8) - q_3(n; 6) + q_2(n; 4) - q_1(n; 2) + 1.$$

Replacing n by 2, 4, 6, 8 in turn, we get

$$-q_1 + 1 = 0,$$
$$q_2 - 6q_1 + 1 = 0,$$
$$-q_3 + 15q_2 - 15q_1 + 1 = 0,$$
$$q_4 - 28q_3 + 70q_2 - 28q_1 + 1 = 0.$$

These relations readily give

$$q_1 = 1,\quad q_2 = 5,\quad q_3 = 61,\quad q_4 = 1385.$$

Thus for each $n > 8$, we have

$$P_n(2, 4, 6, 8) = 1385\,(n; 8) - 61\,(n; 6) + 5\,(n; 4) - (n; 2) + 1.$$

In particular for $n = 9$,

$$P_9(2, 4, 6, 8) = 12465 - 5124 + 630 - 36 + 1 = 7936.$$

16. Permutations with a given Index and a Given Degree

Let

$$F = [f_1, f_2, \ldots, f_n] \text{ where } f\text{'s are nonnegative integers;} \qquad \ldots(16.1)$$

$$J(s) = \{j_1, j_2, \ldots, j_s\} \text{ where } 0 < j_1 < j_2 < \ldots < j_s; \qquad \ldots(16.2)$$

be the degree vector and the index set of any permutation

$$a_1a_2 \ldots a_n \qquad \ldots(16.3)$$

with the given index i and the given degree d, where we assume that the numbers i and d are both positive.

Then, we have

$$f_1 + f_2 + \dots + f_n = d; \quad \dots(16.4)$$

$$j_1 + j_2 + \dots + j_s = i. \quad \dots(16.5)$$

If

$$R(s) = \{r_1, r_2, \dots, r_s\} \quad \dots(16.6)$$

is the interval set corresponding to (16.2), then we also have

$$sr_1 + (s-1)\, r_2 + (s-2) r_3 + \dots + 2r_{s-1} + r_s = i. \quad \dots(16.7)$$

It will be recalled that in (16.6) and (16.7), r's are positive integers.

We first assert that for $i + d = n$, there is at least one permutation (16.3) of S_n with index i, degree d and $a_n < n$.

PROOF : The permutation is, in fact, given by

$$\begin{aligned} a_k &= k \text{ for each } k \leqslant i - 1; \\ &= n \text{ for } k = i; \\ &= k - 1 \text{ for each } k \text{ such that } i < k \leqslant n. \end{aligned}$$

On the other hand for $n > (i + d)$, there is no permutation (16.3) of S_n with index i, degree d and $a_n < n$.

PROOF : If in (16.3), $a_n < n$, then there exists a unique $k \leqslant j_s$ for which $a_k = n$. In view of (16.5), this $k \leqslant i$.

The f_k corresponding to this $k = n - k$.

In view of (16.4), $f_k \leqslant d$. Hence

$$n \leqslant k + d \leqslant i + d.$$

This contradicts the assumption that $n > i + d$, and our assertion follows.

It is thus clear that the permutations with index i and degree d must be looked for among the permutations of S_n with $n \leqslant i + d$. Permitting a_n to have the value n, all the permutations of index i and degree d can be listed if we take $n = i + d$, and this we shall do in all that follows.

Let $A(i, d)$ denote the number of permutations of S_n, $n = i + d$, with index i and degree d. Then, we have merely to find a formula for $M_d(J(s))$—the number of permutations of S_n with degree d and the index set $J(s)$ and $A(i, d)$ will be determined by letting $J(s)$ run over all the partitions of i into one or more distinct summands in (16.5). The number of summands in any such partition cannot exceed h where h is the largest integer for which

$$(h + 1; 2) \leqslant i. \qquad \dots(16.8)$$

In fact, we shall have

$$A(i, d) = \sum_{s \leqslant h} M_d(J(s)) \qquad \dots(16.9)$$

with h as in (16.8).

Determination of $M_d(J(s))$

It is clear that $M_d(J(s))$ is the number of vectors F satisfying eqn. (16.4) and the conditions :

$$f_k > f_{k+1} \text{ if and only if } k = j_1, j_2, \dots, j_s. \qquad \dots(16.10)$$

Let

$$\left.\begin{array}{l} f_1 + \dots + f_{j_1} = d_1, \\ f_{j_1+1} + \dots + f_{j_2} = d_2, \\ \dots\dots\dots\dots\dots\dots \\ f_{j_{s-1}+1} + \dots + f_{j_s} = d_s. \end{array}\right\} \qquad \dots(16.11)$$

Since

$$f_{j_s+1} = f_{j_s+2} = \dots = f_n = 0 \qquad \dots(16.12)$$

we must have

$$d_1 + d_2 + \dots + d_s = d. \qquad \dots(16.13)$$

Here $d_1, d_2, \dots, d_s$ are necessarily positive integers.

Hence d must be greater than s or equal to it.

Noting that relations (16.11) provide partitions of numbers $d_1, d_2, \dots, d_s$ into $r_1, r_2, \dots, r_s$ nonnegative parts respectively, one way of dealing with the problem in any given case will be the following :

List all the partitions of $d_1, d_2, \dots, d_s$ into $r_1, r_2, \dots, r_s$ parts respectively. Take any partition of d_1 from your list. To its right write down one of the partitions of d_2. To the right of this one put down a partition of d_3 and carry on till you have written down a partition of d_s. Now add $(n - j_s)$ 0's to the right. In this manner, you will obtain

$$p(d_1, r_1)\, p(d_2, r_2) \dots p(d_s, r_s)$$

vectors F. Keep those that satisfy each and every one of the conditions in (16.10). The number of these acceptable vectors is the contribution of the solution $(d_1, d_2, \dots, d_s)$ of (16.13) to $M_d(J(s))$. The sum of the contributions of all the $(s - 1; d - 1)$ solutions of (16.13) in positive integers is $M_d(J(s))$ for the given $J(s)$.

A computer can easily handle the problem in this manner.

It is clear that

$$M_d(J(s)) \leqslant \sum_{D^*} p(d_1, r_1)\, p(d_2, r_2) \ldots p(d_s, r_s) \qquad \ldots(16.14)$$

where D^* runs over all the solutions of (16.13) in positive integers. In particular when $J(s)$ is a one element set, we have

$$M_d(\{k\}) = p(d, k). \qquad \ldots(16.15)$$

This shows that

$$M_k(\{k\}) = p(k); \qquad \ldots(16.16)$$

so that

$$A(k, k) \geqslant p(k),\ n \geqslant 2k.$$

This is an improvement on Theorem 1 of Alter *et al.* (1974).

Remark — The results presented in Appendix 2, were obtained in the manner described above but without the aid of a computer.

A formula for $M_d(J(s))$.

In (16.13), $d_1, d_2, \ldots, d_s$ had been allowed to run over all solutions of that equation in positive integers. We now relax this condition and let them run over all solutions of the equation

$$t_1 + t_2 + \ldots + t_s = d \qquad \ldots(16.17)$$

in 'nonnegative' integers. Denote this set of solutions by T^*.

Write

$$W_d(J(s)) = \sum_{T^*} p(d_1, r_1)\, p(d_2, r_2) \ldots p(d_s, r_s)$$

where $p(0, r_k) = 1$ for each $k \leqslant s$.

Then $W_d(J(s))$ counts all those permutations of S_n which have the degree d and whose index sets are non-empty subsets of $J(s)$.

Since for $d > 0$, $M_d(\phi) = 0$, we have

$$W_d(J(s)) = \sum_{J^*} M_d(J^*) \qquad \ldots(16.18)$$

where J^* runs over all the subsets of $J(s)$.

The inclusion and exclusion principle now gives

$$M_d\,(J(s)) = \sum_{J^*} (-1)^{s-|J^*|}\, W_d(J^*). \qquad \text{...(16.19)}$$

Here $W_d(\phi) = 1$, by definition. We are thus through.

Example — $M_5\,(\{1, 3, 6\}) =$

$$W_5(\{1, 3, 6\}) - W_5(\{1, 3\}) - W_5(\{1, 6\}) - W_5(\{3, 6\})$$
$$+ W_5(\{1\}) + W_5(\{3\}) + W_5(\{6\}) - 1$$
$$= \sum_{\substack{T^* \\ s=3 \\ d=5}} p(d_1, 1)\, p(d_2, 2)\, p(d_3, 3)$$
$$- \sum_{\substack{T^* \\ s=2 \\ d=5}} \{p(d_1, 1)\, p(d_2, 2) + p(d_1, 1)\, p(d_2, 5) + p(d_1, 3)\, p(d_2, 3)\}$$
$$+ p(5, 1) + p(5, 3) + p(5, 6) - 1.$$
$$= 58 - 60 + 1 + 5 + 7 - 1 = 10.$$

17. The Generating Function of $W_d(J(s))$

As a direct consequence of our definition of $W_d(J(s))$ in the preceding section, it will be clear that $W_d(J(s)$ is the coefficient of x^d in

$$X_{r_1} X_{r_2} \ldots X_{r_s} \qquad \text{...(17.1)}$$

where

$$X_k = \{(1 - x)\,(1 - x^2) \ldots (1 - x^k)\}^{-1}. \qquad \text{...(17.2)}$$

It will be useful to remark here that

$$X_{r_1} X_{r_2} \ldots X_{r_s} = X_{j_s}\, \{X_{r_1} X_{r_2} \ldots X_{r_s}/X_{j_s}\}. \qquad \text{...(17.3)}$$

The expression within the curly brackets on the right of (17.3) will be seen to be a polynomial in x, with integral coefficients, and of degree at most

$$(j_s + 1;\, 2) - \sum_{k=1}^{s} (r_k + 1;\, 2). \qquad \text{...(17.4)}$$

Observe further that for any $h < k$, we can write

$$X_h = (1 - x^{h+1})\,(1 - x^{h+2}) \ldots (1 - x^k)\, X_k. \qquad \text{...(17.5)}$$

The generating function of $A(u, d)$—the number of permutations of S_n of any given index u and of degree d (any $d > 0$), can now be written out as in the following example with $u = 8$.

Index set	*Generating function*
$\{8\}$	$X_8 - 1$
$\{1, 7\}$	$X_1X_6 - X_1 - X_7 + 1$
$\{2, 6\}$	$X_2X_4 - X_2 - X_6 + 1$
$\{3, 5\}$	$X_3X_2 - X_3 - X_5 + 1$
$\{1, 2, 5\}$	$X_1X_1X_3 - X_1X_1 - X_1X_4 - X_2X_3 + X_1 + X_2 + X_5 - 1$
$\{1, 3, 4\}$	$X_1X_2X_1 - X_1X_2 - X_1X_3 - X_3X_1 + X_1 + X_3 + X_4 - 1.$

Denoting by C_u the generating function of $A(u, d)$, we thus have

$$\begin{aligned} C_8 = X_8 &+ (x + x^2 + x^3 + x^4 + x^5 + x^6)\, X_7 \\ &+ (2 + 2x + 3x^2 + 2x^3 + 3x^4 + 2x^5)\, X_6 \\ &+ (1 + 2x + 2x^2 + 3x^3 + 3x^4)\, X_5 \\ &- (1 + x + 0x^2 - x^3)\, X_4 - (3 + 2x + x^2)\, X_3 \\ &- (1 + x)\, X_2 + X_1 \\ = X_8(x &+ 4x^2 + 7x^3 + 11x^4 + 11x^5 + 10x^6 + 5x^7 - x^8 \\ &- 13x^9 - 21x^{10} - 27x^{11} - 27x^{12} - 21x^{13} - 11x^{14} \\ &+ 2x^{15} + 15x^{16} + 25x^{17} + 27x^{18} + 25x^{19} + 16x^{20} \\ &+ 7x^{21} - 4x^{22} - 11x^{23} - 15x^{24} - 13x^{25} - 9x^{26} \\ &- 4x^{27} + 0x^{28} + 3x^{29} + 5x^{30} + 4x^{31} + 2x^{32} + 0x^{33} \\ &- x^{34} - x^{35}). \end{aligned}$$

Example — $A(8, 8) = p(7) + 4p(6) + 7p(5) + 11p(4) + 11p(3)$
$+ 10p(2) + 5p(1) - p(0)$
$= 15 + 44 + 49 + 55 + 33 + 20 + 5 - 1 = 220.$

This is in agreement with the result of Alter-Curtz-Wang (1974). As an example of the use of (17.5), we give the following :

We have (Appendix 4)

$$C_3 = X_3(x + x^2 + x^3 - x^4 - x^5) \quad \text{...(17.6)}$$

$$\begin{aligned} = X_6(x &+ x^2 + x^3 - x^4 - 2x^5 - 2x^6 - 3x^7 - x^8 + x^9 + 3x^{10} \\ &+ 3x^{11} + 3x^{12} + x^{13} - x^{14} - 2x^{15} - 2x^{16} - x^{17} - x^{18} \\ &+ x^{19} + x^{20}). \end{aligned} \quad \text{...(17.7)}$$

Using (17.6), we will get

$$\begin{aligned} A(3, 6) &= p(5, 3) + p(4, 3) + p(3, 3) - p(2, 3) - p(1, 3) \\ &= 5 + 4 + 3 - 2 - 1 = 9. \end{aligned}$$

From (17.7), we obtain

$$A(3, 6) = p(5) + p(4) + p(3) - p(2) - 2p(1) - 2p(0)$$
$$= 7 + 5 + 3 - 2 - 2 - 2 = 9.$$

The latter gives the result in terms of unrestricted partitions only.

We have thus shown that C_u can be expressed in either of the forms:

$$C_u = X_u(c_{u,1}x + c_{u,2}x^2 + \ldots + c_{u,k}x^k),\ k = (u + 1; 2)$$

or $$C_u = X_v(c'_{v,1}\, x + c'_{v,2}\, x^2 + \ldots + c'_{v,h}\, x^h),\ h = (v + 1; 2)$$

where the c and c' are integers, not necessarily positive or non-zero, and $v > u$.

18. The Alter-Curtz-Wang Conjecture

Alter *et al.* (1974) conjectured that for each $n \leqslant (u + v)$,

$$A(u, v) = A(v, u). \qquad \ldots(18.1)$$

For $n = u + v$, this means that the coefficient of x^v in C_u = the coefficient of x^u in C_v. Assuming, without loss of generality, that $v > u$ and v is not too big, for any given u and v, (18.1) can be verified as follows.

Let

$$C_u = X_v(c'_{v,1}\, x + c'_{v,2}\, x^2 + \ldots + c'_{v,v}\, x^v + \ldots)$$

$$C_v = X_v(c_{v,1}x + c_{v,2}x^2 + \ldots + c_{v,u}x^u + \ldots).$$

Then, all that we have to do is to check if the two expressions

$$c'_{v,1}\, p(v - 1) + c'_{v,2}\, p(v - 2) + \ldots + c'_{v,v}\, p(0)$$

and

$$c_{v,1}p(u - 1) + c_{v,2}p(u - 2) + \ldots + c_{v,u}p(0)$$

are or are not equal.

Example — For $u = 5$, $v = 8$, we have

$$C_5 = X_8(x + 2x^2 + 3x^3 + 2x^4 + x^5 - 3x^6 - 5x^7 - 7x^8 + \ldots)$$

$$C_8 = X_8(x + 4x^2 + 7x^3 + 11x^4 + 11x^5 + \ldots).$$

Now

$$p(7) + 2p(6) + 3p(5) + 2p(4) + p(3) - 3p(2) - 5p(1) - 7p(0)$$
$$= 15 + 22 + 21 + 10 + 3 - 6 - 5 - 7 = 53;$$

and

$$p(4) + 4p(3) + 7p(2) + 11p(1) + 11p(0)$$
$$= 5 + 12 + 14 + 11 + 11 = 53.$$

Hence $A(5, 8) = A(8, 5)$.

We now proceed to prove that the Alter-Curtz-Wang conjecture holds good for each $u \leqslant 4$ and all v.

We make no use of the generating functions. All that we use is the method of direct computation suggested in Section 16. Details are given for the case $u = 3$. For $u = 4$, only the contributions from different interval sets are noted. The cases $u = 1$ and $u = 2$ are simple and are left to the reader.

Proof for $u = 3$

Since the conjecture is easy to verify for small values of v, we take $v > 9$.

We first show that

$$A(3, v) = [(v^2 + 3)/12] + v. \qquad \text{...(18.2)}$$

The contribution of the index set $\{3\}$ to $A(3, v)$ is $p(v, 3)$ and that of the index set $\{1, 2\}$ is the same as the number of partitions of v into two distinct summands.

Since $p(v, 3) = [(v^2 + 3)/12] + [v/2] + 1;$

and the number of partitions of v into two distinct summands is

$$[(v - 1)/2];$$

(18.2) follows at once.

On the other side, let $v = m + 9$.

Then, the contribution of the index set $\{v\}$ to $A(v, 3)$ is $p(3)$. The contribution of the two solutions (1, 2) and (2, 1) of

$$d_1 + d_2 = 3$$

for the index set $\{1, v - 1\}$ is 3, for the remaining index sets except the last one which is $\{(v - 1)/2, (v + 1)/2\}$ when v is odd and $\{(v - 2)/2, (v + 2)/2\}$ when v is even, it is 4. For the last index set it is 1 when v is odd and 3 when v is even.

Hence the total contribution from the two element index sets is

$$2v - 6 \text{ for all } v > 9.$$

Finally, the solution (1, 1, 1) of the equation

$$d_1 + d_2 + d_3 = 3$$

makes a contribution of 1 or 0 to $A(v, 3)$, according as an interval set $\{r_1, r_2, r_3\}$ given by the equation

$$3r_1 + 2r_2 + r_3 = v \qquad \text{...(18.3)}$$

does or does not satisfy the three conditions

$$r_1 \geqslant 1, \ r_2 \geqslant 2, \ r_3 \geqslant 2.$$

The number of interval sets satisfying these conditions is the same as the number of partitions of $v - 9$ into at most three parts or what is the same thing, the number of partitions of $(v - 6)$ into exactly three parts. This is

$$[((v - 6)^2 + 3)/12] = [((m + 3)^2 + 3)/12].$$

Thus, we have

$$A(v, 3) = p(3) + (2v - 6) + [(v^2 - 12v + 39)/12]$$
$$= v + [(v^2 + 3)/12].$$

and we are through.

Proof for $u = 4$

We take $v > 16$.

For $A(4, v)$, the contributions for different interval sets are :

Interval set	*Contribution*
$\{4\}$	$p(v, 4)$
$\{1, 2\}$	$[(v + 4)(v - 1)/6]$.

For $A(v, 4)$, these are

$\{v\}$	$p(4, v) = 5$
$\{r_1, r_2\}$ with $2r_1 + r_2 = v$	$2[(5v - 18)/2]$
$\{r_1, r_2, r_3\}$ with $3r_1 + 2r_2 + r_3 = v$	$(v^2 - 13v + 44)/2$
$\{r_1, r_2, r_3, r_4\}$ with $4r_1 + 3r_2 + 2r_3 + r_4 = v$	$p(v - 16, 4)$

To establish the equality of $A(4, v)$ and $A(v, 4)$, we use the identity

$$p(v, 4) = p(v - 16, 4) + p(v, 3) + p(v - 4, 3) + p(v - 8, 3) + p(v - 12, 3).$$

We might observe here that for any fixed u and v ($> u^2$) sufficiently large but in an arbitrarily fixed residue class modulo the l.c.m. of 1, 2, 3, ..., u; the contribution to $A(u, v)$ from the k-element interval sets is a polynomial in v of degree $(k - 1)$, not necessarily with integral coefficients. In the case of $A(v, u)$, it is the contribution from $(u + 1 - k)$-element interval sets which is a polynomial of degree $(k - 1)$ in v. (The two polynomials are however not usually indentical). In fact, the major contribution to $A(u, v)$ comes from the one-element interval set $\{u\}$ and is given by $p(v, u)$, while the major contribution to $A(v, u)$ comes from the u-element interval sets $\{r_1, r_2, \ldots, r_u\}$ satisfying the relation

$$ur_1 + (u - 1) r_2 + \ldots + r_u = v \qquad \ldots(18.4)$$

from the solution $(1, 1, 1, \ldots, 1)$ of the equation

$$d_1 + d_2 + \ldots + d_u = u.$$

This contribution is the same as the number of solutions of (18.4)

with $\quad r_1 \geqslant 1;\ r_2, r_3, \ldots, r_u$ each $\geqslant 2$;

and is therefore given by $p(v - u^2, u)$.

Since for v sufficiently large and u fixed,

$$p(v - u^2, u) \sim p(v, u),$$

it follows that

$$A(u, v) \sim A(v, u). \qquad \ldots(18.5)$$

This strongly supports the Alter-Curtz-Wang conjecture.

19. *As* AND PANTAGONAL NUMBERS

Assume that $v \geqslant u$, and

$$C_v = X_v(c_{v,1}x + c_{v,2}x^2 + \ldots + c_{v,v}x^v + \ldots).$$

Then, as in section 17, we have

$$\begin{aligned}
A(v, 1) &= c_{v,1}\,p(0)\\
A(v, 2) &= c_{v,1}\,p(1) + c_{v,2}\,p(0)\\
A(v, 3) &= c_{v,1}\,p(2) + c_{v,2}\,p(1) + c_{v,3}\,p(0)\\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
A(v, u) &= c_{v,1}\,p(u - 1) + c_{v,2}\,p(u - 2) + c_{v,3}\,p(u - 3) + \ldots + c_{v,u}\,p(0).
\end{aligned}$$

Now, for each $t > 0$, we have

$$p(t) = \sum_{k \geqslant 1} (-1)^{k-1} \{p(t - \tfrac{1}{2}k(3k - 1)) + p(t - \tfrac{1}{2}k(3k + 1))\}$$

with $p(-q) = 0$ for each $q \geqslant 1$. This means that

$$p(t) - p(t - 1) - p(t - 2) + p(t - 5) + p(t - 7) - p(t - 12) - \ldots = 0$$

where the numbers 1, 2, 5, 7, ... are the pentagonal numbers

$$\tfrac{1}{2}k(3k - 1) \text{ and } \tfrac{1}{2}k(3k + 1).$$

This immediately gives the nice relation :

$$A(v, u) - A(v, u-1) - A(v, u-2) + A(v, u-5) + - - + + - - + + \dots$$
$$= c_{v,u}. \qquad \dots(19.1)$$

Given the c s, this enables us to compute $A(v, 1)$, $A(v, 2)$, ..., $A(v, v)$ in succession without the use of any partition table.

Example — For $v = 8$, we have

$$A(8, 1) = 1;$$

$$A(8, 2) - A(8, 1) = 4, \quad A(8, 2) = 5;$$

$$A(8, 3) - A(8, 2) - A(8, 1) = 7, \quad A(8, 3) = 13;$$

$$A(8, 4) - A(8, 3) - A(8, 2) = 11, \quad A(8, 4) = 29;$$

$$A(8, 5) - A(8, 4) - A(8, 3) = 11, \quad A(8, 5) = 53;$$

$$A(8, 6) - A(8, 5) - A(8, 4) + A(8, 1) = 10, \quad A(8, 6) = 91;$$

$$A(8, 7) - A(8, 6) - A(8, 5) + A(8, 2) = 5, \quad A(8, 7) = 144;$$

$$A(8, 8) - A(8, 7) - A(8, 6) + A(8, 3) + A(8, 1) = -1, \quad A(8, 8) = 220.$$

Note that we take $A(8, 0) = 0$.

Using $c's$ in place of c s, one can compute the values of $A(u, v)$ for any given $v \geqslant u$ in the same manner.

But the Alter-Curtz-Wang conjecture still evades proof except for the few cases we have considered in the preceding section.

Acknowledgement

My thanks are due to Professors Lehmer and Alter for their making a copy of the paper by Alter *et al.* (1974) available to me. My thanks are also due to Drs. J. C. Parnami, R. J. Hans Gill and V. C. Dumir of the Department of Mathematics, Panjab University, Chandigarh, discussions with whom helped in clarifying some of my ideas.

References

Alter, Ronald, Curtz, Thaddeus B., and Wang, Chung C. (1974). Permutations with fixed index and number of inversions. *Proceedings of the S-E Conference on Combinatorics, Graph Theory and Computing*, *X*, 209-28.

Foulkes, H. O. (1976). Enumeration of permutations with prescribed up-down and inversion sequences. *Discrete Math.*, **15**, 235-52.

Gupta, Hansraj, Gwyther, C. E., and Miller, J. C. P. (1958). Tables of Partitions. University Press, Cambridge.

Appendix 1

A complete census of the F vectors of permutations with index 5 and degree 7, and of those with index 7 and degree 5. The zero part at the end is dropped. The summits are indicated by crosses.

1	2	3	4	5	n
				×	
0	0	0	0	7	12
0	0	0	1	6	11
0	0	0	2	5	10
0	0	0	3	4	9
0	0	1	1	5	10
0	0	1	2	4	9
0	0	1	3	3	8
0	0	2	2	3	8
0	1	1	1	4	9
0	1	1	2	3	8
0	1	2	2	2	7
1	1	1	1	3	8
1	1	1	2	2	7
×			×		
1	0	0	6		10
1	0	1	5		9
1	0	2	4		8
1	0	3	3		7
2	0	0	5		9
2	0	1	4		8
2	0	2	3		7
2	1	1	3		7
2	1	2	2		6
3	0	0	4		8
3	0	1	3		7
3	0	2	2		6
3	1	1	2		6
4	0	0	3		7
4	0	1	2		6
4	1	1	1		5
5	0	0	2		6
5	0	1	1		6
6	0	0	1		7
	×	×			
0	6	1			8
0	5	2			7
0	4	3			6
1	5	1			7
1	4	2			6
2	4	1			6
2	3	2			5
3	3	1			5

1	2	3	4	5	6	7	n
						×	
0	0	0	0	0	0	5	12
0	0	0	0	0	1	4	11
0	0	0	0	0	2	3	10
0	0	0	0	1	1	3	10
0	0	0	0	1	2	2	9
0	0	0	1	1	1	2	9
0	0	1	1	1	1	1	8
×					×		
1	0	0	0	0	4		10
1	0	0	0	1	3		9
1	0	0	0	2	2		8
1	0	0	1	1	2		8
1	0	1	1	1	1		7
2	0	0	0	0	3		9
2	0	0	0	1	2		8
2	0	0	1	1	1		7
3	0	0	0	0	2		8
3	0	0	0	1	1		7
4	0	0	0	0	1		7
	×			×			
0	1	0	0	4			9
0	1	0	1	3			8
0	1	0	2	2			7
0	2	0	0	3			8
0	2	0	1	2			7
0	2	1	1	1			6
1	1	0	0	3			8
1	1	0	1	2			7
0	3	0	0	2			7
0	3	0	1	1			6
1	2	0	0	2			7
1	2	0	1	1			6
0	4	0	0	1			6
1	3	0	0	1			6
2	2	0	0	1			6
		×	×				
0	0	4	1				7
0	0	3	2				6
0	1	3	1				6
0	2	2	1				5
1	1	2	1				5
×	×		×				
3	1	0	1				5
2	1	0	2				6

$n = 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12$

$A = 3 + 9 + 10 + 8 + 5 + 3 + 1 + 1 = 40$

Appendix 2

The following calculations support the Alter-Curtz-Wang conjecture :

(i) $d = 6, \quad i = 10 \quad n \geqslant 16 \quad d = 10, \quad i = 6$

Index set	Contribution	Index set	Contribution
10	11	6	35
1, 9	18	1, 5	63
2, 8	32	2, 4	62
3, 7	36		
4, 6	24	1, 2, 3	4
1, 2, 7	7		164
1, 3, 6	22		
1, 4, 5	7		
2, 3, 5	7		
	164		

(ii) $d = 10, \quad i = 7 \quad n \geqslant 17 \quad d = 7, \quad i = 10$

7	38	10	15
1, 6	77	1, 9	29
2, 5	111	2, 8	53
3, 4	30	3, 7	61
		4, 6	40
1, 2, 4	22	1, 2, 7	14
		1, 3, 6	40
	278	1, 4, 5	13
		2, 3, 5	13
		1, 2, 3, 4	0
			278

(iii) $d = 6, \quad i = 11 \quad n \geqslant 17 \quad d = 11, \quad i = 6$

11	11	6	44
1, 10	18	1, 5	83
2, 9	32	2, 4	79
3, 8	38		
4, 7	34	1, 2, 3	5
5, 6	8		
1, 2, 8	7		211
1, 3, 7	24		
1, 4, 6	22		
2, 3, 6	10		
2, 4, 5	7		
	211		

APPENDIX 3

Generating functions C_u, for values of $u \leqslant 7$

$u = 1 \quad X_1 - 1$

$u = 2 \quad X_2 - 1$

$u = 3 \quad X_3 + xX_2 - X_1$

$u = 4 \quad X_4 + (x + x^2) X_3 - X_1$

$u = 5 \quad X_5 + (x + x^2 + x^3) X_4 + (x + x^2) X_3 - X_2 - X_1 + 1$

$u = 6 \quad X_6 + (x + x^2 + x^3 + x^4) X_5 + (1 + x + 2x^2 + x^3) X_4 - (2 + x) X_2$

$u = 7 \quad X_7 + (x + x^2 + x^3 + x^4 + x^5) X_6 + (1 + 2x + 2x^2 + 2x^3 + 2x^4) X_5$
$\quad + (x + x^2 + 2x^3) X_4 - (2 + x) X_3 - (1 + x) X_2 + 1.$

APPENDIX 4

Generating functions C_u in expanded form

$u = 1 \quad X_1 x$

$u = 2 \quad X_2(x + x^2 - x^3)$

$u = 3 \quad X_3(x + x^2 + x^3 - x^4 - x^5)$

$u = 4 \quad X_4(x + 2x^2 + x^3 + x^4 - 2x^5 - 2x^6 - x^7 + 0x^8 + x^9)$

$u = 5 \quad X_5(x + 2x^2 + 3x^3 + 2x^4 + x^5 - 3x^6 - 4x^7 - 4x^8 - 2x^9$
$\quad + x^{10} + 2x^{11} + 2x^{12} + x^{13} + 0x^{14} - x^{15})$

$u = 6 \quad X_6(x + 3x^2 + 4x^3 + 4x^4 + 2x^5 + x^6 - 5x^7 - 7x^8 - 7x^9$
$\quad - 5x^{10} - 2x^{11} + 2x^{12} + 5x^{13} + 4x^{14} + 3x^{15} + x^{16}$
$\quad + 0x^{17} - 2x^{18} - x^{19})$

$u = 7 \quad X_7(x + 3x^2 + 6x^3 + 7x^4 + 7x^5 + 3x^6 + 0x^7 - 8x^8 - 13x^9$
$\quad - 15x^{10} - 13x^{11} - 8x^{12} + 0x^{13} + 7x^{14} + 11x^{15}$
$\quad + 13x^{16} + 9x^{17} + 5x^{18} - x^{19} - 4x^{20} - 5x^{21} - 4x^{22}$
$\quad - 3x^{23} + 0x^{24} + x^{25} + 2x^{26} + x^{27} - x^{28}).$

Expressions for $u = 8$ have already been given in the text.