ON MULTIPLE FINITE MATHIEU TRANSFORMS

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The aim of the present investigation is to develop multiple finite Fourier Mathieu transforms which do not seem available in the literature. While solving the multi-dimensional wave equation or the diffusion equation the method of Laplace transform sometimes becomes very long and often involves the evaluation of intricate contour integrals. In many situations the method of finite Fourier Mathieu transforms is quicker and simpler to use. The importance of the transforms developed in this paper has been verified by solving the diffusion equation.

1. INTRODUCTION

The Mathieu functions $ce_p(x, q)$ and $se_p(x, q)$ (Mclachlan 1964, p. 21) of integral order p are the solutions of the Mathieu's equation :

$$y'' + (\alpha - 2q \cos 2x)\, y = 0 \qquad ...(1)$$

where the eigenvalues α are the functions of q.

The preferred form of the series is given below for q positive :

$$ce_p(x, q) = \sum_{r=0,1}^{\infty}{}^{*} A_r^{(p)} \cos rx \qquad ...(2)$$

$$se_p(x, q) = \sum_{r=0,1}^{\infty}{}^{*} B_r^{(p)} \sin rx \qquad ...(3)$$

where the asterisk '*' over the summation sign indicates that the summation is taken over only even or odd values of r as r is even or odd.

In these series A and B are functions of q. Generally, the series appear to converge absolutely and uniformly if q is small, but diverge when it is large enough. Moreover, these functions are continuous for all real x.

Hence, by virtue of property of Sturm-Liouville systems (Margenan and Murphy 1943), the functions $ce_p(x, q)$ and $se_p(x, q)$ are complete on the interval $(0, \pi)$ with respect to all square integrable functions.

For analogy and convenience, the Mathieu functions shall appear simultaneously like $\begin{smallmatrix} ce_p \\ se_p \end{smallmatrix}(x, q)$ throughout the present paper.

2. Multiple Finite Mathieu Transforms

2.1. *Inversion Theorem for Triple Fourier Mathieu Transforms*

If $f(x, y, z)$ and its first partial derivatives are piecewise continuous over the cube $\{(x, y, z) : 0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi\}$, and if triple finite Fourier Mathieu transform of $f(x, y, z)$ is defined by the equation

$$T_{mnp}[(f(x, y, z)\ ;\ (x, y, z) \to (q, r, s)] \equiv \bar{f}_{mnp}(q, r, s)$$

$$= \int_0^\pi \int_0^\pi \int_0^\pi f(x, y, z) \begin{smallmatrix} ce_m \\ se_m \end{smallmatrix}(x, q) \begin{smallmatrix} ce_n \\ se_n \end{smallmatrix}(y, r) \begin{smallmatrix} ce_p \\ se_p \end{smallmatrix}(z, s)\, dx\, dy\, dz \qquad \ldots(4)$$

then

$$T^{-1}_{mnp}[\bar{f}_{mnp}(q, r, s)\ ;\ (q, r, s) \to (x, y, z)] = f(x, y, z)$$

$$= \left(\frac{2}{\pi}\right)^3 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \bar{f}_{mnp}(q, r, s) \begin{smallmatrix} ce_m \\ se_m \end{smallmatrix}(x, q) \begin{smallmatrix} ce_n \\ se_n \end{smallmatrix}(y, r) \begin{smallmatrix} ce_p \\ se_p \end{smallmatrix}(z, s) \qquad \ldots(5)$$

at points of the cube at which the function $f(x, y, z)$ is continuous, the modification required when it has finite discontinuity is obvious.

PROOF : By generalized Fourier series (Churchill 1963, p. 148), we let

$$f(x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} D_{mnp} \begin{smallmatrix} ce_m \\ se_m \end{smallmatrix}(x, q) \begin{smallmatrix} ce_n \\ se_n \end{smallmatrix}(y, r) \begin{smallmatrix} ce_p \\ se_p \end{smallmatrix}(z, s),$$

$$(0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant \pi,\ 0 \leqslant z \leqslant \pi). \qquad \ldots(6)$$

By grouping terms in this triple Mathieu series so as to display the total coefficients of $ce_m(x, q)$ $[se_m(x, q)]$ for each m, we can write, formally,

$$f(x, y, z) = \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} \sum_{p=0}^{\infty} D_{mnp} \begin{smallmatrix} ce_n \\ se_n \end{smallmatrix}(y, r) \begin{smallmatrix} ce_p \\ se_p \end{smallmatrix}(z, s) \right] \begin{smallmatrix} ce_m \\ se_m \end{smallmatrix}(x, q) \qquad \ldots(7)$$

Now normalizing the function $ce_m(x, q)$ $[se_m(x, q)]$ by stipulation (Mclachlan 1964, p. 24) so that

$$\frac{2}{\pi}\int_0^\pi ce_m^2(x,q)\,dx = \frac{2}{\pi}\int_0^\pi se_m^2(x,q)\,dx = 1 \qquad \ldots(8)$$

for all real values of q (m is even or odd), we get

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} D_{mnp}\,\frac{ce_n}{se_n}(y,r)\,\frac{ce_p}{se_p}(z,s) = \frac{2}{\pi}\int_0^\pi f(x,y,z)\,\frac{ce_m}{se_m}(x,q)\,dx$$

$$= F_m(y,z), \text{ say.} \qquad \ldots(9)$$

The right-hand side of (9) is a sequence of functions $F_m(y,z)$, $m = 0, 1, \ldots,$ each represented by its Fourier series (9) on the square $\{(y,z) : 0 \leqslant y \leqslant \pi, 0 \leqslant z \leqslant \pi\}$, where the coefficients

$$\sum_{p=0}^{\infty} D_{mnp}\,\frac{ce_p}{se_p}(z,s)$$

of $ce_n(y,r)$ $[se_n(y,r)]$ can be determined in a similar fashion to give us

$$\sum_{p=0}^{\infty} D_{mnp}\,\frac{ce_p}{se_p}(z,s) = \frac{2}{\pi}\int_0^\pi F_m(y,z)\,\frac{ce_n}{se_n}(y,r)\,dy$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^\pi\int_0^\pi f(x,y,z)\,\frac{ce_m}{se_m}(x,q)\,\frac{ce_n}{se_n}(y,r)\,dx\,dy$$

$$= F_{mn}(z), \text{ say.} \qquad \ldots(10)$$

Again, repeating the same procedure, we get

$$D_{mnp} = \frac{2}{\pi}\int_0^\pi F_{mn}(z)\,\frac{ce_p}{se_p}(z,r)\,dz$$

$$= \left(\frac{2}{\pi}\right)^3 \int_0^\pi\int_0^\pi\int_0^\pi f(x,y,z)\,\frac{ce_m}{se_m}(x,q)\,\frac{ce_n}{se_n}(y,r)\,\frac{ce_p}{se_p}(z,r)\,dx\,dy\,dz. \qquad \ldots(11)$$

Thus, by virtue of (11) and (6), the inversion formula (5) follows.

Corollary — If we are dealing with a problem in which $f(x,y,z)$ is defined over the region $\{(x,y,z) : 0 < x < a, 0 < y < b, 0 < z < c\}$, we have following modified inversion theorem :

If $f(x, y, z)$ and its first partial derivatives are piecewise continuous over the region $\{(x, y, z) : 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b, 0 \leqslant z \leqslant c\}$, and if finite Fourier Mathieu transform of $f(x, y, z)$ is defined by the equation

$$T_{mnp}\,[f(x, y, z)\ ;\ (x, y, z) \to (q, r, s)] \equiv \bar{f}_{mnp}(q, r, s)$$

$$= \int_0^a \int_0^b \int_0^c f(x, y, z) \begin{matrix} ce_m \\ se_m \end{matrix}\left(\frac{\pi x}{a}, q\right) \begin{matrix} ce_n \\ se_n \end{matrix}\left(\frac{\pi y}{b}, r\right) \begin{matrix} ce_p \\ se_p \end{matrix}\left(\frac{\pi z}{c}, s\right) dx\, dy\, dz \quad \text{...(12)}$$

then

$$T^{-1}_{mnp}\,[\bar{f}_{mnp}(q, r, s)\ ;\ (q, r, s) \to (x, y, z)] \equiv f(x, y, z)$$

$$= \frac{8}{abc} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \bar{f}_{mnp}(q, r, s) \begin{matrix} ce_m \\ se_m \end{matrix}\left(\frac{\pi x}{a}, q\right) \begin{matrix} ce_n \\ se_n \end{matrix}\left(\frac{\pi y}{b}, r\right) \begin{matrix} ce_p \\ se_p \end{matrix}\left(\frac{\pi z}{c}, s\right) \quad \text{...(13)}$$

at points of the region at which the function $f(x, y, z)$ is continuous ; the modification required when it has finite discontinuity is obvious.

The above transforms of functions of three variables can be extended to the functions of N-variables.

2.2. *Particular Cases*

If $q \to 0$, $r \to 0$, $s \to 0$, then $ce_m(x, q) \to \cos(mx)$, $ce_n(y, r) \to \cos(ny)$, $ce_p(z, s) \to \cos(pz)$, $se_m(x, q) \to \sin(mx)$, etc. Therefore, the transforms developed in this paper reduce to well-known multiple finite Fourier cosine and sine transforms (Sneddon 1951).

3. Applications

As an example of applications of the transforms developed in this paper, we solve the diffusion equation :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - 2\,[q \cos 2x + r \cos 2y + s \cos 2z]\, u = K \frac{\partial u}{\partial t} \quad \text{...(14)}$$

where q, r, s are positive constants, under the boundary conditions :

(*i*) $u = 0$, on the boundary of $\{(x, y, z)\ ;\ 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b, 0 \leqslant z \leqslant c\}$ ...(15)

(*ii*) $u = f(x, y, z)$, initially. ...(16)

From the examination of boundary conditions (15), (16), the appropriate transform to be used here is obviously with the kernel

$$se_m\left(\frac{\pi x}{a}, q\right) se_n\left(\frac{\pi y}{b}, r\right) se_p\left(\frac{\pi z}{c}, r\right).$$

Thus, applying Mathieu transform (4) on both sides of equation (14), we get

$$\int_0^a \int_0^b \int_0^c \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - 2(q \cos 2x + r \cos 2y + s \cos 2z)u\right]$$

$$\times\ se_m\left(\frac{\pi x}{a}, q\right) se_n\left(\frac{\pi y}{b}, r\right) se_p\left(\frac{\pi z}{c}, s\right) dx\, dy\, dz$$

$$= K \int_0^a \int_0^b \int_0^c \frac{\partial u}{\partial t} se_m\left(\frac{\pi x}{a}, q\right) se_n\left(\frac{\pi y}{b}, r\right) se_p\left(\frac{\pi z}{c}, s\right) dx\, dy\, dz. \quad \ldots(17)$$

This equation reduces to the simple linear differential equation :

$$-[\alpha(q) + \beta(r) + \gamma(z)]\, \bar{u} = K\frac{\partial \bar{u}}{\partial t}, \text{ (see Appendix)} \quad \ldots(18)$$

whose solution is given by

$$\bar{u} = A\, e^{-[\alpha(q)+\beta(r)+\gamma(s)]t/K}. \quad \ldots(19)$$

Since $\bar{u} = \bar{f}_{mnp}(q, r, s)$ initially ($t = 0$), the constant A thus becomes equal to $\bar{f}_{mnp}(q, r, s)$. Hence

$$\bar{u} = \bar{f}_{mnp}(q, r, s)\, e^{-[\alpha(q)+\beta(r)+\gamma(s)]t/K} \quad \ldots(20)$$

Now applying the inversion formula (13), we get the desired solution

$$u = \frac{8}{abc} e^{-Q} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} se_m\left(\frac{\pi x}{a}, q\right) se_n\left(\frac{\pi y}{b}, r\right) se_p\left(\frac{\pi z}{c}, s\right)$$

$$\times\ \bar{f}_{mnp}(q, r, s), \quad \ldots(21)$$

where, for convenience, $Q = [\alpha(q) + \beta(r) + \gamma(s)]\, t/K$; m, n, and p are even or odd.

Remark : In case, we change the boundary condition (15) to

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$$

on the boundary of $\{(x, y, z)\ ;\ 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b,\ 0 \leqslant z \leqslant c\}$, the appropriate transform to be used will be one with the kernel

$$ce_m\left(\frac{x\pi}{a}, q\right) ce_n\left(\frac{y\pi}{b}, r\right) ce_p\left(\frac{z\pi}{c}, s\right).$$

Similarly, the use of other six similar transforms obtained by interchanging *ce* by *se* in (4) will also depend upon the nature of respective boundary conditions.

4. Conclusions

The beauty of the transforms developed in this paper lies in the fact that they do facilitate the resolution of boundary value problems by providing an operational technique which can be used to reduce the governing partial differential equation to simple differential equation. Since the analysis of boundary value problems in elasticity often needs to know the transforms of fourth order differential operators, the results can be deduced by successive applications of the results obtained for second order differential operators.

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Appendix

Theorem — If $f(x, y, z)$ and its first partial derivatives are bounded in the region $R = \{(x, y, z) : 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b,\ 0 \leqslant z \leqslant c\}$, then triple finite Fourier Mathieu transform of $Lf(x, y, z)$ exists, and is given by

$$T_{mnp}[L f(x, y, z)] = -[\alpha(q) + \beta(r) + \gamma(s)]\, T_{mnp}[f(x, y, z)] \qquad \ldots(A)$$

where

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - 2(q \cos 2x + r \cos 2y + s \cos 2z)$$

provided $f(x, y, z)$ and its first partial derivatives vanish on the boundary of R.

Proof : By definition (4) we first evaluate

$$T_{mnp}\left[\frac{\partial^2 f}{\partial x^2} - 2qf\cos 2x\right]$$

$$= \int_0^a \int_0^b \int_0^c \frac{\partial^2 f}{\partial x^2} \begin{matrix} ce_m \\ se_m \end{matrix}\left(\frac{\pi x}{a}, q\right) \begin{matrix} ce_n \\ se_n \end{matrix}\left(\frac{\pi y}{b}, r\right) \begin{matrix} ce_p \\ se_p \end{matrix}\left(\frac{\pi z}{c}, s\right) dx\, dy\, dz$$

$$- 2q \int_0^a \int_0^b \int_0^c f(x, y, z) \cos 2x \begin{matrix} ce_m \\ se_m \end{matrix}\left(\frac{\pi x}{a}, q\right) \begin{matrix} ce_n \\ se_n \end{matrix}\left(\frac{\pi y}{b}, q\right) \begin{matrix} ce_p \\ se_p \end{matrix}\left(\frac{\pi z}{c}, q\right) \times dx\, dy\, dz$$

$$= I_1 + I_2, \text{ say.}$$

Now, evaluating x-integral of I_1 twice by parts, we get

$$I_1 = \int_0^b \int_0^c \begin{matrix} ce_n \\ se_n \end{matrix}\left(\frac{\pi y}{b}, r\right) \begin{matrix} ce_p \\ se_p \end{matrix}\left(\frac{\pi z}{c}, s\right)$$

$$\times \left[\frac{\partial f}{\partial x} \begin{matrix} ce_m \\ se_m \end{matrix}\left(\frac{\pi x}{a}, q\right) - f(x, y, z) \frac{\partial}{\partial x}\left(\begin{matrix} ce_m \\ se_m \end{matrix}\left(\frac{\pi x}{a}, q\right)\right)\right]_0^a dy\, dz$$

$$+ \int_0^a \int_0^b \int_0^c \begin{matrix} ce_n \\ se_n \end{matrix}\left(\frac{\pi y}{b}, r\right) \begin{matrix} ce_p \\ se_p \end{matrix}\left(\frac{\pi z}{c}, s\right) \frac{\partial^2}{\partial x^2}\left(\begin{matrix} ce_m \\ se_m \end{matrix}\left(\frac{\pi x}{a}, q\right)\right)$$

$$\times f(x, y, z)\, dx\, dy\, dz.$$

Since first term of I_1 vanishes because of the prescribed boundary conditions, we have

$$T_{mnp}\left[\frac{\partial^2 f}{\partial x^2} - 2qf\cos 2x\right]$$

$$= \int_0^a \int_0^b \int_0^c f(x, y, z) \begin{matrix} ce_n \\ se_n \end{matrix}\left(\frac{\pi y}{b}, r\right) \begin{matrix} ce_p \\ se_p \end{matrix}\left(\frac{\pi z}{c}, s\right)$$

$$\times \left[\frac{\partial^2}{\partial x^2}\left(\begin{matrix} ce_m \\ se_m \end{matrix}\left(\frac{\pi x}{a}, q\right)\right) - 2q\cos 2x \begin{matrix} ce_m \\ se_m \end{matrix}\left(\frac{\pi x}{a}, q\right)\right] dx\, dy\, dz$$

$$= -\alpha(q)\, T_{mnp} f(x, y, z)$$

where $\alpha(q)$ are the eigenvalues of x-Mathieu's equation. Thus by symmetry we obtain (A), where $\beta(r)$ and $\gamma(s)$ are the eigenvalues of the corresponding Mathieu's equations.